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OLIY

MATEMATIKA

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OLIY MATEMATIKA FANIDAN AMALIY MASHG‘ULOTLAR O‘TKAZISHGA DOIR O‘QUV QO‘LLANMA

IKKI JILDLIK

1-JILD

TATU va uning xududiy filiallari talabalari uchun

TOSHKENT 2016

SO‘Z BOSHI

Ushbu o‘quv qo‘llanma O‘zR Oliy va o‘rta maxsus ta’lim vazirligi tomonidan tasdiqlangan “Oliy matematika,” fanining o‘quv rejasiga to‘la mos keladi va bu o‘quv qo‘llanma bakalavriatning quyidagi ta’lim yo‘nalishi talabalariga mo‘ljallangan:

5330500 – Kompyuter injiniringi (“Kompyuter injiniringi”, “AT-servis”, “Multimedia texnologiyalari”);
5330300 – Axborot xavfsizligi;
5330600 – Dasturiy injiniring;
5350100 – Telekommunikatsiya texnologiyalari (“Telekommunikatsiyalar”, “Teleradioeshittirish”, Mobil tizimlari);
5350200 –Televizion texnologiyalar (“Audiovizual texnologiyalar”, “Telestudiya tizimlari va ilovalari”);
5350300 – Axborot-kommunikatsiya texnologiyalari sohasida iqtisodiyot va menejment;
5350400 – Axborot-kommunikatsiya texnologiyalari sohasida kasb ta’limi;
5350500 –Pochta aloqasi texnologiyasi;
5350600 –Axborotlashtirish va kutubxonashunoslik.

O‘quv qo‘llanma ikki jilddan iborat bo‘lib, uning birinchi jildida chiziqli algebra, tekslikda va fazoda analitik geometriya, vektorlar algebrasi elementlari, matematik analizga kirish, bir o‘zgaruvchili funksiyalarning differensial hisobi, funksiyalarni hosilalar yordamida tekshirish, bir o‘zgaruvchili funksiyalarning integral hisobi kiritilgan. Har bir paragrafda dastlab qisqacha nazariy ma’lumotlar keltirilib, keyin esa turli tipdagi misol va masalalarning batafsil yechilish usullar ko‘rsatilib, kerakli uslubiy ko‘rsatmalar berilgan. Har bir bo‘lim uchun mustaqil yechish uchun yetarli miqdorda misol va masalalar hamda test savollari berilgan. Undan tashqari har bir bo‘limda berilayotgan nazariy bilimlarni amaliyot bilan bog‘lovchi masalalar yechib ko‘rsatilgan va mustaqil bajarish uchun topshiriqlar berilgan.

Kitob hajmini ixchamlashtirish maqsadida unda quyidagi belgilashlar kiritilgan:
►- masala va misollar yechilishining boshlanishi;
◄- masala va misollar yechilishining tugallanishi.

Mazkur qo‘llanma yaratishda mualliflar o‘zlarining Toshkent axborot texnologiyalari universitetining talabalariga ko‘p yillar mobaynida o‘qilgan ma’ruzalar va talabalar bilan o‘tkazilgan amaliy mashg‘ulotlarni asos qilib olingan, shuningdek mavjud adabiyotlardan ham foydalanilgan. O‘quv qo‘llanma kamchiliklardan holi emas, albatta. Qo‘llanmadagi kamchiliklarni bartaraf etishga va uning sifatini yaxshilashga qaratilgan fikr va mulohazalarini bildirganlarga mualliflar avvaldan o‘z minnatdorchliklarini bildiradilar.

MUNDARIJA

I BOB CHIZIQLI ALGEBRA ELEMENTLARI

1.1 Determinantlar va ularning xossalari. Determinantlarni hisoblash usullari

Ikkinchi tartibli determinant deb
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \qquad (1.1)$$
tenglik bilan aniqlanadigan songa aytiladi. Qisqacha, Δ deb belgilanadi. Bu yerda a_{11}, a_{12}, a_{21}, a_{22} -determinantning elementlari deyiladi. .

a_{11}, a_{12} va a_{21}, a_{22} mos ravishda determinantning 1- va 2-satrlari, a_{11}, a_{21} va a_{12}, a_{22} mos ravishda determinantning 1- va 2-ustunlari deyiladi. Ya'ni
$$a_{ij}: \begin{cases} i - \text{satr tartibi} \\ j - \text{ustun tartibi.} \end{cases}$$
Determinantning ixtiyoriy satri yoki ustuni determinantning *qatori* deb ataladi. a_{11}, a_{22} -elementlar joylashgan diagonal *bosh diagonal* deyiladi. a_{21}, a_{12} -elementlar joylashgan diagonal *yordamchi diagonal* deyiladi.

1-misol

Hisoblang: $\begin{vmatrix} 3 & 2 \\ -4 & 5 \end{vmatrix}$

► (1.1) formulani qo'llaymiz:
$$\begin{vmatrix} 3 & 2 \\ -4 & 5 \end{vmatrix} = 3\cdot 5 - 2\cdot(-4) = 15 + 8 = 23. \blacktriangleleft$$

Eslatma Determinantning elementlari funksiyalar bo'lishi ham mumkin, shuning uchun determinantning qiymati, umuman olganda, funksiyadir.

2-misol

Hisoblang: $\begin{vmatrix} \cos x & \sin x \\ \sin x & \cos x \end{vmatrix}$.

► $\begin{vmatrix} \cos x & \sin x \\ \sin x & \cos x \end{vmatrix} = \cos^2 x - \sin^2 x = \cos 2x$. ◄

Uchinchi tartibli determinant deb
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \qquad (1.2)$$
tenglik bilan aniqlanadigan songa aytiladi. Ko'pincha, determinant tartibiga mos ravishda Δ_3 deb ham belgilanadi.

3-misol

Hisoblang: $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 4 \\ 2 & -3 & 5 \end{vmatrix}$.

► (1.2) formulani qo‘llaymiz:

$$\begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 4 \\ 2 & -3 & 5 \end{vmatrix} = 2\cdot1\cdot5+(-1)\cdot4\cdot2+3\cdot1\cdot(-3)-3\cdot1\cdot2-(-1)\cdot1\cdot5-2\cdot4\cdot(-3)=$$

$$=10-8-9-6+5+24=16.$$ ◄

Determinantning a_{ij} elementining M_{ij} *minori* deb, uning $i-$ satri va $j-$ ustunini o‘chirishdan hosil bo‘lgan determinantga aytiladi.

Masalan, uchunchi tartibli determinant uchun

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

Determinantning a_{ij} elementining A_{ij} *algebraik to‘ldiruvchisi* deb,

$$A_{ij} = (-1)^{i+j} M_{ij}$$

tenglik bilan aniqlanadigan songa aytiladi.

Masalan, uchunchi tartibli determinant uchun

$$A_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \quad A_{22} = (-1)^{2+2} M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}.$$

4-misol

Quyidagi $\begin{vmatrix} 1 & 1 & 4 \\ -1 & 2 & 3 \\ -3 & 2 & 5 \end{vmatrix}$ determinantning M_{23} minorini hisoblang

► Determinantning $2-$ satri va $3-$ ustunini o‘chiramiz:

$$\begin{vmatrix} 1 & 1 & 4 \\ -1 & 2 & 3 \\ -3 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -3 & 2 \end{vmatrix} = 1\cdot2-1\cdot(-3)=2+3=5.$$ Demak, $M_{23}=5$. ◄

5-misol

Quyidagi $\begin{vmatrix} 2 & 1 & -4 \\ 1 & 3 & 5 \\ 3 & 2 & -1 \end{vmatrix}$ determinantning A_{32} va A_{13} algebraik to‘ldiruvchilarini hisoblang.

► $A_{32}=(-1)^{3+2}M_{32}$, ya'ni $A_{32}=-M_{32}$ bo'lgani uchun, determinantning $3-$ satri va $2-$ ustunini o'chiramiz:

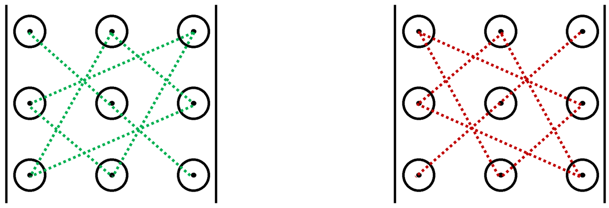
$$A_{32}=-\begin{vmatrix}2 & 1 & -4\\ 1 & 3 & 5\\ 3 & 2 & -1\end{vmatrix}=-\begin{vmatrix}2 & -4\\ 1 & 5\end{vmatrix}=-(2\cdot 5-(-4)\cdot 1)=-14\ .$$

$A_{13}=(-1)^{1+3}M_{32}$ yoki $A_{13}=M_{13}$ bo'lgani uchun, determinantning $1-$ satri va $3-$ ustunini o'chirib hisoblaymiz.

$$A_{13}=\begin{vmatrix}1 & 3\\ 3 & 2\end{vmatrix}=1\cdot 2-3\cdot 3=-7.$$

Demak, $A_{32}=-14,\quad A_{13}=-7$. ◄

Determinant hisoblashning (1.2) formulasini eslab qolish uchun quyidagi sxemani keltiramiz:



Hisoblashning bu qoidasi *uchburchak usuli (Sarryus usuli)* deyiladi. Qulaylik uchun deteminantning birinchi va ikkinchi ustunini quyidagicha parallel ko'chirib, bosh diagonal va yordamchi diagonalga parallel chiziqlar bo'yicha ko'paytmalar tuzamiz:

$$\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}=\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}\begin{matrix}a_{11} & a_{12}\\ a_{21} & a_{22}\\ a_{31} & a_{32}\end{matrix}-\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}\begin{matrix}a_{11} & a_{12}\\ a_{21} & a_{22}\\ a_{31} & a_{32}\end{matrix}=$$

$$=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}\ , \tag{1.3}$$

bunda bosh diagonal bo'yicha hosil qilinga qo'shuluvchilar musbat ishora bilan, yordamchi diagonal bo'yicha hosil qilingan qo'shiluvchilar manfiy ishora bilan olinadi.

6-misol

Quyidagi $\begin{vmatrix}1 & 1 & 4\\ -1 & 2 & 3\\ -3 & 2 & 5\end{vmatrix}$ determinantni hisoblang

► Determinantning birinchi va ikkinchi ustunini parallel ko'chirib yozib Sarryus usulida hisoblaymiz:

$$\begin{vmatrix}1 & 1 & 4\\ -1 & 2 & 3\\ -3 & 2 & 5\end{vmatrix}\begin{matrix}1 & 1\\ -1 & 2\\ -3 & 2\end{matrix}=1\cdot 2\cdot 5+1\cdot 3\cdot(-3)+4\cdot(-1)\cdot 2-4\cdot 2\cdot(-3)-1\cdot 3\cdot 2-1\cdot(-1)\cdot 5=$$

$$=10-9-8+24-6+5=16.◄$$

(1.2) formulani algebraik to‘lduruvchilar yordamida quyidagicha ifodalaymiz:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \qquad (1.4)$$

7-misol

Quyidagi $\begin{vmatrix} 2 & 5 & -4 \\ 1 & -3 & 5 \\ 3 & 2 & -1 \end{vmatrix}$ determinantni hisoblang

► (1.4) formulani qo‘llaymiz, buning uchun avval A_{11}, A_{12} va A_{13} larni hisoblaymiz:

$$A_{11} = \begin{vmatrix} -3 & 5 \\ 2 & -1 \end{vmatrix} = 3-10=-7, \quad A_{12} = -\begin{vmatrix} 1 & 5 \\ 3 & -1 \end{vmatrix} = -(-1-15)=16,$$

$$A_{13} = \begin{vmatrix} 1 & -3 \\ 3 & 2 \end{vmatrix} = 2-(-9)=11.$$

$$\begin{vmatrix} 2 & 5 & -4 \\ 1 & -3 & 5 \\ 3 & 2 & -1 \end{vmatrix} = 2\cdot(-7)+5\cdot16-4\cdot11=-14+80-44=22.◄$$

Determinantning xossalari:

1. *Determinantning barcha satrlarini mos ustunlari bilan almashtirish natijasida qiymati o‘z garmaydi*

2. *Determinantning biror qatoridagi barcha elementlari nolga teng bo‘lsa, uning qiymati nolga teng bo‘ladi.*

3. *Determinantning ikkita parrallel qatorining o‘rinlarini o‘zaro almashtirish natijasida determinant qiymatining ishorasi qarama-qarshisiga o‘zgaradi.*

4. *Determinantning ikkita parrallel qatori bir xil bo‘lsa, uning qiymati nolga teng bo‘ladi.*

5. *Agar determinantning biror qatori bir xil ko‘paytuvchiga ega bo‘lsa, bu ko‘paytuvchini determinant belgisidan tashqariga chiqarish mumkin. Demak, determinantni biror songa ko‘paytirish uchun uning biror qatori elementlarini shu songa ko‘paytirish kifoya.*

6. *Determinantning ikkita parrallel qatori mos pavishda proporsional bo‘lsa, uning qiymati nolga teng bo‘ladi.*

7. *Determinantning qiymati uning biror qatori elementlarini mos algebraik to‘ldiruvchilariga ko‘paytirilib qo‘shilganiga teng.*

8. *Agar determinantning biror qator elementlari yig‘indilardan iborat bo‘lsa, u holda bu determinant ikki determinant yig‘indisiga teng bo‘ladi, bunda birinchi determinantda shu qator birinchi qo‘shuluvchilardan, ikkinchisida esa ikkinchi qo‘shuluvchilardan tashkil topgan bo‘ladi.*

Masalan,

$$\begin{vmatrix} a_{11}+b_1 & a_{12} & a_{13} \\ a_{21}+b_2 & a_{22} & a_{23} \\ a_{31}+b_3 & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{33} \\ b_3 & a_{32} & a_{33} \end{vmatrix}.$$

9. *Agar determinantning biror qatori elementlarini ixtiyoriy songa ko‘paytirib, parallel qatori elementlariga mos ravishda qo‘shilsa, determinant qiymati o‘zgarmaydi.*

10. *Determinantning biror qatori elementlarini parallel qator mos elementlarining algebraik to‘ldiruvchilariga ko‘paytmalari yig‘indisi nolga teng.*

Masalan, $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$.

9-xossa yordamida, (1.4) formuladan ko‘ra umumiyroq bo‘lgan, determinantni *biror qatori bo‘yicha yoyib hisoblash usuli* hosil bo‘ladi. Masalan, uchunchi tartibli determinant uchun

$$\Delta_3 = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3}, \tag{1.5}$$

$$\Delta_3 = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j}. \tag{1.6}$$

Bu yerda (1.5) va (1.6) formulalar mos ravishda determinantning ixtiyoriy $i-$ *satri* va $j-$ *ustuni bo‘yicha yoyilmasi* deyiladi.

8-misol

Quyidagi $\begin{vmatrix} 4 & -2 & 0 \\ 3 & 5 & 6 \\ -3 & 4 & 0 \end{vmatrix}$ determinantni biror qatori bo‘yicha yoyib hisoblang

► Determinantni eng ko‘p nol element qatnashgan qatorini aniqlaymiz. Bu yerda uchunchi ustunda eng ko‘p nol element bo‘lgani uchun, (1.6) formulani qo‘llaymiz: $\Delta_3 = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} = 6A_{23}$, chunki $a_{13} = a_{33} = 0$.

$$\begin{vmatrix} 4 & -2 & 0 \\ 3 & 5 & -6 \\ -3 & 4 & 0 \end{vmatrix} = -6\cdot(-1)^{2+3}\begin{vmatrix} 4 & -2 \\ -3 & 4 \end{vmatrix} = 6\cdot(16-6) = 60. \blacktriangleleft$$

Quyida biz n-tartibli determinantning ko‘rinishini keltiramiz:

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Quyida biz, asosan, yuqori tartibli determinantlarni hisoblashda keng qo‘llanadigan ikkita hisoblash usulini keltiramiz.

1. Yuqori tartibli determinantni, asosiy xossalaridan foydalanib, biror qatorining bitta elementidan boshqa barcha elementlarini nolga aylantirilib, so‘ng 9-xossa yordamida *tartibini pasaytirib hisoblash* mumkin.

9-misol

Quyidagi $\Delta = \begin{vmatrix} 2 & -4 & 1 & 5 \\ 1 & -3 & 2 & 5 \\ 2 & 2 & 0 & -3 \\ 3 & -1 & 1 & 2 \end{vmatrix}$ determinantni hisoblang.

► Determinantning birinchi satrini -2 va -1 ga ko‘paytirib, mos ravishda ikkinchi va to‘rtinchi satriga qo‘shamiz:

$$\Delta = \begin{vmatrix} 2 & -4 & 1 & 5 \\ -3 & 5 & 0 & -5 \\ 2 & 2 & 0 & -3 \\ 1 & 3 & 0 & -3 \end{vmatrix}$$

$3-$ ustun bo‘yicha yoyib(9-xossa), ya’ni $\Delta = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} + a_{43}A_{43}$ va $a_{23} = a_{33} = a_{43} = 0$ ekanini e’tiborga olib, tartibini pasaytiramiz. Hosil bo‘lgan uchunchi tartibli determinantni esa uchburchak usulida yechamiz.

$$\Delta = 1 \cdot (-1)^{1+3} \begin{vmatrix} -3 & 5 & -5 \\ 2 & 2 & -3 \\ 1 & 3 & -3 \end{vmatrix} = 18 - 15 - 30 + 10 + 30 - 27 = -14 . ◄$$

2. Bosh diagonalidan yuqorisidagi yoki pastidagi barcha elementlari nollardan iborat bo‘lgan determinant *uchburchak shaklidagi determinant* deyiladi. Bunday determinantning qiymati bosh diagonali elementlari ko‘paytmasiga teng. Har qanday determinantni *uchburchak shakliga keltirib hisoblash* mumkin.

10-misol

Quyidagi $\Delta = \begin{vmatrix} 2 & -4 & 1 & 5 \\ 1 & -3 & 2 & 5 \\ 2 & 2 & 0 & -3 \\ 3 & -1 & 1 & 2 \end{vmatrix}$ determinantni uchburchak shakliga keltirib hisoblang.

► Determinantning a_{11} elementini 1 ga aylantirish uchun birinchi va ikkinchi satrlarining o‘rinlarini almashtiramiz. Hosil bo‘lgan birinchi satrni -2, -2 va -3 ga ko‘paytirib, mos ravishda ikkinchi , uchunchi va to‘rtinchi satrlarga qo‘shamiz.

$$\Delta = -\begin{vmatrix} 1 & -3 & 2 & 5 \\ 2 & -4 & 1 & 5 \\ 2 & 2 & 0 & -3 \\ 3 & -1 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & -3 & 2 & 5 \\ 0 & 2 & -3 & -5 \\ 0 & 8 & -4 & -13 \\ 0 & 8 & -5 & -13 \end{vmatrix}$$

Uchinchi satrini -1 ga ko‘paytirib to‘rtinchi satrlarga qo‘shamiz:

$$\Delta=-\begin{vmatrix}1 & -3 & 2 & 5\\ 0 & 2 & -3 & -5\\ 0 & 8 & -4 & -13\\ 0 & 0 & -1 & 0\end{vmatrix}$$

Ikkinchi satrini -4 ga ko‘paytirib uchinchi satriga qo‘shamiz. So‘ng uchinchi va to‘rtinchi satrlar o‘rinlarini almashtiramiz:

$$\Delta=-\begin{vmatrix}1 & -3 & 2 & 5\\ 0 & 2 & -3 & -5\\ 0 & 0 & 8 & 7\\ 0 & 0 & -1 & 0\end{vmatrix}=\begin{vmatrix}1 & -3 & 2 & 5\\ 0 & 2 & -3 & -5\\ 0 & 0 & -1 & 0\\ 0 & 0 & 8 & 7\end{vmatrix}$$

Uchinchi satrini 8 ga ko‘paytirib to‘rtinchi satriga qo‘shamiz :

$$\Delta=\begin{vmatrix}1 & -3 & 2 & 5\\ 0 & 2 & -3 & -5\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 7\end{vmatrix}=1\cdot 2\cdot(-1)\cdot 7=-14\cdot$$ ◄

Auditoriya topshiriqlari

1. Berilgan ikkinchi tartibli deteminantlarni hisoblang.

a) $\begin{vmatrix}5 & 3\\ 7 & -4\end{vmatrix}$; b) $\begin{vmatrix}4 & -7\\ -2 & -3\end{vmatrix}$; d) $\begin{vmatrix}tgx & -1\\ 1 & tgx\end{vmatrix}$.

2. Tenglamani yeching.

a) $\begin{vmatrix}x+3 & 2\\ 7 & x-2\end{vmatrix}=0$; b) $\begin{vmatrix}\sin 2x & -\cos 2x\\ \sin 3x & \cos 3x\end{vmatrix}=0$

3. Berilgan uchinchi tartibli deteminantlarni hisoblang.

a) $\begin{vmatrix}1 & -2 & 4\\ -3 & 5 & 5\\ 2 & -1 & 3\end{vmatrix}$; b) $\begin{vmatrix}10 & -2 & 4\\ -15 & 3 & 6\\ 20 & -1 & 5\end{vmatrix}$; d) $\begin{vmatrix}1 & 2 & 5\\ 5 & -3 & 7\\ 4 & 6 & 5\end{vmatrix}$.

4. Berilgan uchinchi tartibli deteminantlarni satr yoki ustun bo‘yicha yoyib hisoblang.

a) $\begin{vmatrix}5 & 0 & 6\\ 4 & 0 & 5\\ 2 & 4 & 3\end{vmatrix}$; b) $\begin{vmatrix}2 & 2 & -1\\ 7 & 0 & 3\\ 3 & 4 & 0\end{vmatrix}$, d) $\begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 0\end{vmatrix}$.

5. Berilgan deteminantlarni uchburchak shakliga keltirib hisoblang.

a) $\begin{vmatrix}2 & -3 & 2 & 4\\ -3 & 2 & 2 & 5\\ 1 & 5 & -3 & 0\\ 0 & -1 & 1 & 2\end{vmatrix}$; b) $\begin{vmatrix}4 & 0 & -3 & 5\\ 3 & 2 & -2 & 1\\ 1 & 3 & 1 & 0\\ 5 & 6 & 2 & -1\end{vmatrix}$.

6. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ deteminantlarni $a-b$, $a-c$ va $b-c$ larga bo‘linishini isbotlang.

7. $\begin{vmatrix} 1 & 6 & 9 \\ 2 & 3 & 4 \\ 3 & 2 & 5 \end{vmatrix}$ deteminantni hisoblamasdan, 13 ga bo‘linishini isbotlang.

Mustaqil yechish uchun testlar

1. To‘g‘ri tengliklarni aniqlang

1) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} d & c \\ b & a \end{vmatrix}$, 2) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$, 3) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} c & d \\ a & b \end{vmatrix}$, 4) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} b & a \\ d & c \end{vmatrix}$.

A) $1),3)$; B) $1),2)$; D) $2),3)$; E) $3),4)$.

2. $\begin{vmatrix} 2 & 1 & 2 \\ -2 & 3 & 0 \\ 1 & 0 & 2 \end{vmatrix}$ determinantning a_{21} elementining M_{21} minorini toping:

A) 4 B) -4 D) 2 E) -2.

3. $\begin{vmatrix} 2 & 1 & 2 \\ -2 & 3 & 0 \\ 1 & 0 & 2 \end{vmatrix}$ determinantning a_{21} elementining A_{21} algebraik to‘ldiruvchisiini toping:

A) 4 B) -4 D) 2 E) -2.

4. $\begin{vmatrix} 0 & 3 & 7 \\ 1 & -3 & 4 \\ 0 & 2 & 6 \end{vmatrix}$ deteminantni hisoblang

A) 4 B) -4 D) 2 E) -2.

5. Agar n-tartibli determinantning satrlarini teskari tartibda yozib chiqilsa qiymati qanday o‘zgaradi?

A) $(-1)^n$ ga ko‘payadi; B) $(-1)^{n-1}$ ga ko‘payadi; D) $(-1)^{\frac{n(n-1)}{2}}$ ga ko‘payadi;
E) o‘zgarmaydi.

1.2 Matritsalar va ular ustida amallar. Teskari matritsa

Berilgan m ta satr va n ta ustundan iborat to‘g‘ri burchakli ushbu

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \text{ yoki } A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \tag{2.1}$$

jadvalga $m\times n$ o‘lcovli *matritsa* deyiladi. Bu yerda a_{ij} $(i=1, 2,\dots,m;\ j=1, 2,\dots,n)$ - matritsaning elementlari deyiladi. Matritsalar lotin alifbosidagi bosh harflar bilan belgilanadi. Ba’zan, o‘lchamlarini ifodalash uchun $A_{m\times n}$ kabi belgilanadi.

Matritsalar qisqacha,

$$A=\left(a_{ij}\right)\ (i=\overline{1,m};\ j=\overline{1,n}) \text{ yoki } A=\left\|a_{ij}\right\|\ (i=\overline{1,m};\ j=\overline{1,n}) \tag{2.2}$$

ko‘rinishda ham yoziladi.

Agar matritsada $i=1$ bo‘lsa, bunday

$$A=\left[a_{11} \quad a_{12} \quad \dots \quad a_{1n}\right]$$

matritsa *satr matritsa* deyiladi.

Agar matritsada $j=1$ bo‘lsa, bunday

$$A=\begin{bmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{n1} \end{bmatrix}$$

matritsa *ustun matritsa* deyiladi.

Matritsada $m=n$ bo‘lsa, *kvadrat matritsa* deyiladi:

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}. \tag{2.3}$$

Bosh diagonalidagi elementlari birlardan va qolgan elementlari nollardan iborat bo‘lgan kvadrat matritsa *birlik matritsa* deyiladi va E deb belgilanadi. Masalan,

$$E=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Matritsaning barcha elementlari nollardan iborat bo‘lsa, *nol matritsa* deyiladi va Q deb belgilanadi.

Mos elementlari teng, ya’ni $a_{ij}=b_{ij}$ bo‘lgan bir xil o‘lchamli A va B matritsalar *teng matritsalar* deyiladi.

Matritsaning satrlarini mos ustunlariga almashtirishdan hosil bo‘lgan matritsa *transponirlangan matritsa* deyiladi va A^T kabi belgilanadi.

Masalan, $A=\begin{bmatrix} 3 & 2 \\ -4 & 5 \end{bmatrix}$ matritsa berilgan bo‘lsa, A^T matritsani hisoblash uchun satrlarini mos ustunlariga almashtiramiz: $A^T=\begin{bmatrix} 3 & -4 \\ 2 & 5 \end{bmatrix}$.

Bir xil o‘lchovli A va B matritsalarni qo‘shish va ayirish mumkin.

Bir xil o‘lchovli *A va B matritsalarni yig‘indisi(ayirmasi)* deb shunday C matritsaga aytiladiki, uning elementlari A va B matritsalarning mos elementlari yig‘indisiga(ayirmasiga) teng. $C=A+B$ ($C=A-B$) kabi belgilanadi.

A matritsani λ songa ko‘paytmasi deb, barcha elementlarini λ songa ko‘paytirishdan hosil bo‘lgan B matritsaga aytiladi, $B=\lambda A$ kabi belgilanadi.

Matritsalarni qo‘shish va songa ko‘paytirish quyidagi xossalarga ega:

i. $A+B=B+A$

ii. $A+Q=A$

iii. $\lambda(A+B)=\lambda A+\lambda B$

iv. $(\lambda+\mu)A=\lambda A+\mu A$

1-misol

Quyida $A=\begin{bmatrix} -2 & 5 \\ 3 & -1 \end{bmatrix}$, $B=\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$ matritsalar berilgan bo‘lsa $A+B$ va $2A-B$ matritsalarni hisoblang

► $A+B=\begin{bmatrix} -2 & 5 \\ 3 & -1 \end{bmatrix}+\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}=\begin{bmatrix} -2+3 & 5+(-1) \\ 3+2 & -1+4 \end{bmatrix}=\begin{bmatrix} 1 & 4 \\ 5 & 3 \end{bmatrix}$;

$$2A-B=2\cdot\begin{bmatrix} -2 & 5 \\ 3 & -1 \end{bmatrix}-\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}=\begin{bmatrix} 2\cdot(-2) & 2\cdot 5 \\ 2\cdot 3 & 2\cdot(-1) \end{bmatrix}-\begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix}=$$

$$=\begin{bmatrix} -4-3 & 10-(-1) \\ 6-2 & -2-3 \end{bmatrix}=\begin{bmatrix} -7 & 11 \\ 4 & -5 \end{bmatrix}.$$ ◄

2-misol

Quyida $A=\begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & 5 \end{bmatrix}$ $B=\begin{bmatrix} -3 & 1 \\ -5 & 4 \\ 0 & 2 \end{bmatrix}$ matritsalar berilgan. A^T+2B va $A-B^T$ matritsalarni hisoblang

► $A^T+B=\begin{bmatrix} 2 & 4 \\ -1 & 0 \\ 3 & 5 \end{bmatrix}+2\cdot\begin{bmatrix} -3 & 1 \\ -5 & 4 \\ 0 & 2 \end{bmatrix}=\begin{bmatrix} 2+2\cdot(-3) & 4+2\cdot 1 \\ -1+2(-5) & 0+2\cdot 4 \\ 3+2\cdot 0 & 5+2\cdot 2 \end{bmatrix}=\begin{bmatrix} -4 & 6 \\ -11 & 8 \\ 3 & 9 \end{bmatrix}$;

$$A-B^T=\begin{bmatrix}2 & -1 & 3\\ 4 & 0 & 5\end{bmatrix}-\begin{bmatrix}-3 & -5 & 0\\ 1 & 4 & 2\end{bmatrix}=\begin{bmatrix}2-(-3) & -1-(-5) & 3-0\\ 4-1 & 0-4 & 5-2\end{bmatrix}=\begin{bmatrix}5 & 4 & 3\\ 3 & -4 & 3\end{bmatrix}. \blacktriangleleft$$

Berilgan $m\times k$ o‘lchovli A matritsani $k\times n$ o‘lchovli B matritsaga *ko‘paytmasi* deb, shunday $m\times n$ o‘lchovli C matritsaga aytiladiki, uning elementlari

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{ik}b_{kj} \tag{2.4}$$

tenglik bilan aniqlanadi. $C=A\cdot B$ kabi belgilanadi.

Demak, birinchi matritsaning ustunlari soni ikkinchi matritsaning satrlari soniga teng bo‘lgan holdagini ularni ko‘paytirish mumkin. Umuman olganda, $A\cdot B$ ko‘paytma mavjud bo‘ganda $B\cdot A$ ko‘paytma mavjud bo‘lavermaydi. $B\cdot A$ ko‘paytma mavjud bo‘gan holda ham, umuman olganda, $AB\neq BA$.

Agar $A\cdot B=B\cdot A$ bo‘lsa, A va B matritsalar *kommutativ* matritsalar deyiladi.

3-misol

Quyida $A=\begin{bmatrix}3 & 1 & 2\\ -2 & 0 & 5\end{bmatrix}$, $B=\begin{bmatrix}1 & 2\\ -5 & 3\\ 0 & 4\end{bmatrix}$ matritsalar berilgan. $A\cdot B$ va $B\cdot A$ ko‘paytmalarni hisoblang

► Bu yerda $A_{2\times3}$ va $B_{3\times2}$ bo‘lgani uchun AB matritsa 2×2 o‘lchovli bo‘ladi:

$$A\cdot B=\begin{bmatrix}3 & 1 & 2\\ -2 & 0 & 5\end{bmatrix}\cdot\begin{bmatrix}1 & 2\\ -5 & 3\\ 0 & 4\end{bmatrix}=\begin{bmatrix}3\cdot1+1\cdot(-5)+2\cdot0 & 3\cdot2+1\cdot3+2\cdot4\\ -2\cdot1+0\cdot(-5)+5\cdot0 & -2\cdot2+0\cdot3+5\cdot4\end{bmatrix}=$$

$$=\begin{bmatrix}-2 & 17\\ -2 & 16\end{bmatrix}.$$

$B\cdot A$ matritsa esa 3×3 o‘lchovli bo‘ladi:

$$B\cdot A=\begin{bmatrix}1 & 2\\ -5 & 3\\ 0 & 4\end{bmatrix}\cdot\begin{bmatrix}3 & 1 & 2\\ -2 & 0 & 5\end{bmatrix}=\begin{bmatrix}1\cdot3+2\cdot(-2) & 1\cdot1+2\cdot0 & 1\cdot2+2\cdot5\\ -5\cdot3+3\cdot(-2) & -5\cdot1+3\cdot0 & -5\cdot2+3\cdot5\\ 0\cdot4+4\cdot(-2) & 0\cdot1+4\cdot0 & 0\cdot2+4\cdot5\end{bmatrix}=$$

$$=\begin{bmatrix}-1 & 1 & 12\\ -21 & -5 & 5\\ -8 & 0 & 20\end{bmatrix}. \quad A\cdot B\neq B\cdot A. \blacktriangleleft$$

4-misol

Quyida $A=\begin{bmatrix}2 & 2\\ 1 & 2\end{bmatrix}$, $B=\begin{bmatrix}3 & 4\\ 2 & 3\end{bmatrix}$ matritsalar berilgan. $A\cdot B$ va $B\cdot A$ ko‘paytmalarni hisoblang

► $A\cdot B=\begin{bmatrix}2&2\\1&2\end{bmatrix}\cdot\begin{bmatrix}3&4\\2&3\end{bmatrix}=\begin{bmatrix}2\cdot3+2\cdot2&2\cdot4+2\cdot3\\1\cdot3+2\cdot2&1\cdot4+2\cdot3\end{bmatrix}=\begin{bmatrix}10&14\\7&10\end{bmatrix};$

$B\cdot A=\begin{bmatrix}3&4\\2&3\end{bmatrix}\cdot\begin{bmatrix}2&2\\1&2\end{bmatrix}=\begin{bmatrix}3\cdot2+4\cdot1&3\cdot2+4\cdot2\\2\cdot2+3\cdot1&2\cdot2+3\cdot2\end{bmatrix}=\begin{bmatrix}10&14\\7&10\end{bmatrix}.$

$A\cdot B=B\cdot A$, demak, A va B matritsalar kommutativlanadigan matritsalar.◄

Matritsalarni ko‘paytirish quyidagi xossalarga ega:

i. $(\lambda A)B=\lambda(AB)$

ii. $(A+B)C=AC+AB$

iii. $A(B+C)=AB+AC$

iv. $A(BC)=(AB)C$

Transponirlangan matritsa uchun esa quyidagi formulalar o‘rinli:

1. $\left(A^T\right)^T=A$
2. $(AB)^T=B^T\cdot A^T$

Agar A kvadrat matritsaning determinanti noldan farqli bo‘lsa, ya’ni $\det A\neq 0$ bo‘lsa, A matritsa *xosmas matritsa* deyiladi.

Agar $\det A=0$ bo‘lsa, A matritsa *xos matritsa* deyiladi.

Agar $AA^{-1}=A^{-1}A=E$ tenglik o‘rinli bo‘lsa, A^{-1} matritsa A xosmas matrtsaning *teskari matritsasi* deyiladi. Bu yerda E matritsa A matritsa o‘lchovi bilan bir xil o‘lchovli birlik matritsadir.

Xosmas matritsa A uchun yagona A^{-1} teskari matritsa mavjud va quyidagi formula bilan hisoblanadi:

$$A^{-1}=\frac{1}{\det A}\begin{pmatrix}A_{11}&A_{21}&\ldots&A_{n1}\\A_{12}&A_{22}&\ldots&A_{n2}\\\ldots&\ldots&\ldots&\ldots\\A_{1n}&A_{2n}&\ldots&A_{nn}\end{pmatrix}\qquad(2.5)$$

Teskari matritsa quyidagi xossalarga:

1. $\det\left(A^{-1}\right)=\dfrac{1}{\det A}$
2. $(AB)^{-1}=B^{-1}\cdot A^{-1}$

5-misol

Quyidagi matritsalarning teskarilarini toping

a) $A=\begin{bmatrix}-1&2\\3&4\end{bmatrix}$, b) $A=\begin{bmatrix}2&-4&1\\1&-5&3\\1&-1&1\end{bmatrix}$

► a) $\det A=\begin{vmatrix}-1&2\\3&4\end{vmatrix}=-10\neq 0$ algebraik to‘ldiruvchilarni hisoblaymiz:

$$A_{11}=4,\ A_{12}=-3,\ A_{21}=-2,\ A_{22}=-1$$

Natijada, (2.5) formulaga ko‘ra,

$$A^{-1}=-\frac{1}{10}\begin{bmatrix}4 & -2\\ -3 & -1\end{bmatrix}=\begin{bmatrix}-0,4 & 0,2\\ 0,3 & 0,1\end{bmatrix}$$

Tekshirish:

$$AA^{-1}=\begin{bmatrix}-1 & 2\\ 3 & 4\end{bmatrix}\cdot\begin{bmatrix}-0,4 & 0,2\\ 0,3 & 0,1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=E$$

a) Uchunchi tartibli determinantni hisoblaymiz, $\det A=-8$ va algebraik to‘ldiruvchilar: $A_{11}=-2$, $A_{12}=2$, $A_{13}=4$, $A_{21}=3$, $A_{22}=1$, $A_{23}=-2$, $A_{31}=-7$, $A_{32}=-5$, $A_{33}=-6$. U holda,

$$A^{-1}=-\frac{1}{8}\begin{pmatrix}-2 & 3 & -7\\ 2 & 1 & -5\\ 4 & -2 & -6\end{pmatrix}\ ◄$$

Auditoriya topshiriqlari

1. A va B matritsalar berilgan. $A+B$, $2A-B$ va $A+3B$ matritsalarni toping

a) $A=\begin{bmatrix}3 & 0 & -1\\ 5 & 4 & -2\end{bmatrix}$, $B=\begin{bmatrix}-1 & 3 & 4\\ 0 & 2 & 5\end{bmatrix}$;

b) $A=\begin{bmatrix}2 & -4\\ 1 & -1\\ 5 & -3\end{bmatrix}$, $B=\begin{bmatrix}-3 & 2\\ -2 & 4\\ 0 & 5\end{bmatrix}$

2. A va B matritsalar berilgan. AB va BA matritsalarni toping

a) $A=\begin{bmatrix}1 & -2 & 4\\ -3 & 5 & 0\\ 2 & -1 & 3\end{bmatrix}$, $B=\begin{bmatrix}3 & 2 & 5\\ 4 & 0 & -1\\ 1 & 1 & -3\end{bmatrix}$;

b) $A=\begin{bmatrix}3 & 2 & -1\\ 7 & 4 & -2\end{bmatrix}$, $B=\begin{bmatrix}0 & -4\\ 1 & -1\\ 5 & -3\end{bmatrix}$;

d) $A=\begin{bmatrix}2 & -5 & 3 & 0\end{bmatrix}$, $B=\begin{bmatrix}4\\ 2\\ -3\\ 5\end{bmatrix}$;

e) $A=\begin{bmatrix}11 & 10\\ 2 & 9\end{bmatrix}$, $B=\begin{bmatrix}3 & 5\\ 1 & 2\end{bmatrix}$.

3. A, B va C matritsalar berilgan. $(AB)C = A(BC)$ ekanini tekshiring

$$A=\begin{bmatrix}5 & 3\\ 2 & -1\\ 4 & 7\end{bmatrix},\quad B=\begin{bmatrix}4 & -3\\ -5 & 2\end{bmatrix},\quad C=\begin{bmatrix}3 & 5\\ 1 & 2\end{bmatrix}.$$

4. Berilgan A matritsaning A^{-1} teskari matritsasini toping

a) $A=\begin{bmatrix}3 & -1 & 3\\ 2 & -1 & 2\\ 2 & 1 & 3\end{bmatrix}$;

b) $A=\begin{bmatrix}2 & 4 & -1\\ 5 & 3 & 4\\ 3 & 7 & 0\end{bmatrix}$

Mustaqil yechish uchun testlar

1. $A=\begin{pmatrix}5 & 2\\ 1 & 4\end{pmatrix}$ va $B=\begin{pmatrix}-3 & 1\\ 5 & -2\end{pmatrix}$ matritsa berilgan, $A+2B$ matritsani toping

$A)\begin{pmatrix}-1 & 4\\ 11 & 0\end{pmatrix}$, $B)\begin{pmatrix}-5 & 1\\ 17 & -7\end{pmatrix}$, $C)\begin{pmatrix}-1 & 4\\ 8 & 0\end{pmatrix}$, $D)\begin{pmatrix}-11 & 4\\ 11 & 0\end{pmatrix}$

2. $K=\begin{pmatrix}1 & -2 & 3\\ -2 & 3 & -4\end{pmatrix}$ matritsa berilgan bo‘lsa, $2K$ matritsani toping

$A)\begin{pmatrix}2 & -4 & 6\\ -2 & 3 & -4\end{pmatrix}$, $B)\begin{pmatrix}1 & -2 & 3\\ -4 & 6 & -8\end{pmatrix}$, $C)\begin{pmatrix}2 & -4 & 6\\ -4 & 6 & -8\end{pmatrix}$, $D)$ A va B to'g'ri

3. $A=\begin{pmatrix}5 & 2\\ 1 & 4\end{pmatrix}$ va $B=\begin{pmatrix}-3 & 1\\ 5 & -2\end{pmatrix}$ matritsalar berilgan, $A\cdot B$ matritsani toping

$A)\begin{pmatrix}-13 & 7\\ 25 & -7\end{pmatrix}$, $B)\begin{pmatrix}-5 & 7\\ 8 & 10\end{pmatrix}$, $C)\begin{pmatrix}-5 & 1\\ 17 & -7\end{pmatrix}$, $D)\begin{pmatrix}-10 & 2\\ 5 & -2\end{pmatrix}$

4. Teskari matritsani topish formulasini ko‘rsating?

A) $A^{-1}=\dfrac{1}{\det A}\begin{pmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}$, B) $A^{-1}=\dfrac{1}{\det A}\begin{pmatrix}A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{pmatrix}$

C) $A^{-1}=\begin{pmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}$, D) $A^{-1}=\begin{pmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}$

5. $A=\begin{pmatrix}3 & 2 & 0\\ 1 & 3 & 1\\ 5 & 3 & 0\end{pmatrix}$ bo‘lsa, A^{-1} teskari matritsani toping

A) $A^{-1}=-\begin{pmatrix}-3 & 5 & -12\\ 0 & 0 & 1\\ 2 & -3 & 7\end{pmatrix}$, *B*) $A^{-1}=\begin{pmatrix}-3 & 5 & -12\\ 0 & 0 & 1\\ 2 & -3 & 7\end{pmatrix}$

C) $A^{-1}=\begin{pmatrix}-3 & 0 & 2\\ 5 & 0 & -3\\ -12 & 1 & 7\end{pmatrix}$ *D*) $A^{-1}=\begin{pmatrix}-3 & 0 & 2\\ -5 & 0 & 3\\ -12 & -1 & 7\end{pmatrix}$

1.3 Matritsa rangi. Chiziqli algebraik tenglamalar sistemasi. Kroniker-Kopelli teoremasi

1.3.1 Matritsaning rangi

To‘g‘ri burchakli (xususiy holda kvadrat) A matritsa berilgan bo‘lsin. Uning biror k ta satr va k ta ustunini ajratamiz, kesishmada turgan elementlardan $k-$tartibli determinanat hosil qilamiz. Bu determinant matritsaning *k -tartibli minori* deb ataladi.

Masalan, ushbu $A=\begin{bmatrix} 1 & 1 & 7 & 1 \\ 2 & 3 & 4 & -2 \\ -1 & 6 & 2 & 5 \end{bmatrix}$ matritsaning 2-tartibli minorlaridan biri $M_2=\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}=3-2=1$ bo‘ladi. 3-tartibli minorlaridan biri $M_3=\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -2 \\ -1 & 6 & 5 \end{vmatrix}=15+12+2+3-10+12=34$ bo‘ladi. Berilgan matritsaning 18 ta *2-tartibli*, 4 ta *3-tartibli* minori bor.

Matritsaning rangi deb, uning noldan farqli minorlarining eng yuqori tartibiga aytiladi, *rangA* yoki $r(A)$ kabi belgilanadi.

Matritsada elementar almashtirishlar deb, quyidagi almashtirishlarga aytiladi:

a) Nollardan iborat qatorlarni o‘chirish.
b) Ikkita parallel qatorlarni o‘rnini almashtirish.
c) Bir qatorning barcha elementlarini biror songa ko‘paytirib, boshqa qatorning mos elementlariga qo‘shish.
d) Qatorning barcha elementlarini noldan farqli bir xil songa ko‘paytirish

Bu almashtirishlar natijasida hosil bo‘lgan matritsa berilgan matritsaga *ekvivalent matritsa* deyiladi va $A\sim B$ kabi belgilanadi.

Teorema. *Matritsalar ustida elementar almashtirishlar natijasida uning rangi o‘zgarmaydi.*

Matritsaning rangini 2 xil usulda topish mumkin.

1-usul. *O‘rab turuvchi minorlar usuli*

Bu usulda birinchi noldan farqli k *-tartibli* minori topiladi. k –tartibli minorni o‘z ichiga oluvchi barcha $k+1$ tartibli minorlar ***o‘rab turuvchi minorlar*** deyiladi. k –tartibli minor noldan farqli bo‘lib, bu minorni o‘rab turuvchi barcha $k+1$ tartibli minorlar nolga teng bo‘lganda, matritsaning rangi shu noldan farqli minor tartibiga teng bo‘ladi. Bu usul hisoblash ishlarini ancha kamaytirish imkoniyatini beradi. Agar o‘rab turuvchi $k+1$ tartibli minorlardan birortasi nolga teng bo‘lmasa, ana shu minorni o‘rab turuvchi minorlarni tekshirilib, bu jaroyon davom ettiriladi.

2-usul. *Zinasimon usul* (*yoki elementar almashtirishlar usuli*)

Bu usulda elementar almashtirishlar yordamida matritsa uchburchakli matritsa ko‘rinishiga keltiriladi. Natijada hosil bo‘lgan matritsaning noldan farqli satrlari soni matritsaning rangiga teng bo‘ladi.

1-misol

Berilgan $A=\begin{bmatrix} 1 & 2 & -1 & 1 & 2 \\ 2 & 4 & 4 & -2 & 0 \\ 1 & 2 & -7 & 5 & 6 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix}$ matritsa rangini ikki xil usulda aniqlang.

► *1-usul.* $M_2=\begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix}=4+2=6\neq 0$. Bu minorni o‘rab turuvchi 3-tartibli minorlar soni 6 ta(umumiy holda, 3-tartibli minorlari 40 ta).

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 4 \\ 1 & 2 & -7 \end{vmatrix}=-28+8-4+4-8+28=0,\quad \begin{vmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ 1 & -7 & 5 \end{vmatrix}=20+2-14-4-14+10=0,$$

$$\begin{vmatrix} 1 & -1 & 2 \\ 2 & 4 & 0 \\ 1 & -7 & 6 \end{vmatrix}=24-28-8+12=0,\quad \begin{vmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ 3 & 9 & -5 \end{vmatrix}=-20+6+18-12+18-10=0,$$

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 4 \\ 3 & 6 & 9 \end{vmatrix}=36+24-12+12-24-36=0,\quad \begin{vmatrix} 1 & -1 & 2 \\ 2 & 4 & 0 \\ 3 & 9 & -2 \end{vmatrix}=-8+36-24-4=0$$

Demak, berilgan matritsa uchun $rangA=2$ bo‘ladi.

2-usul. Quyidagicha elementar almashtirishlar olib boramiz: 1)1-satr elementlarini $-2,-1,-3$ larga ko‘paytirib, mos ravishda 2-, 3-, 4-satr elementlariga qo‘shamiz; 2) 2-satr elementlarini $1,-2$ larga ko‘paytirib, mos ravishda 3-, 4-satr elementlariga qo‘shamiz; 3) nollardan iborat satrlarini o‘chiramiz.

$$A=\begin{bmatrix} 1 & 2 & -1 & 1 & 2 \\ 2 & 4 & 4 & -2 & 0 \\ 1 & 2 & -7 & 5 & 6 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix}\sim\begin{bmatrix} 1 & 2 & -1 & 1 & 2 \\ 0 & 0 & 6 & -4 & -4 \\ 0 & 0 & -6 & 4 & 4 \\ 0 & 0 & 12 & -8 & -8 \end{bmatrix}\sim\begin{bmatrix} 1 & 2 & -1 & 1 & 2 \\ 0 & 0 & 6 & -4 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\sim$$

$\sim\begin{bmatrix} 1 & 2 & -1 & 1 & 2 \\ 0 & 0 & 6 & -4 & -4 \end{bmatrix}$. Demak, $rangA=2$ ekan.◄

Tartibi matritsa rangiga teng bo‘lgan minor *bazis minor* deb ataladi. Kesishmasida bazis minor elementlari turgan satrlar va ustunlar *bazis satrlar va ustunlar* deyiladi. Matritsaning istalgan satri(ustuni) uning bazis satrlarining (ustunlarining) chiziqli kombinatsiyasidan iborat bo‘ladi. Bazis satrlar (ustunlar) chiziqli erkli satrlar(ustunlar) bo‘ladi.

Teorema. *Agar matritsaning rangi r ga teng bo‘lsa, u holda unda r ta chiziqli erkli satr topiladi, qolgan barcha satrlar esa bu* r *ta satrning chiziqli kombinatsiyasi bo‘ladi.*

Natija. *Matritsaning rangi undagi chiziqli erkli satrlar(ustunlar) soniga teng.*

1.3.2 Chiziqli algebraik tenglamalar sistemasi

Quyidagi umumiy ko‘rinishdagi n ta noma’lumli m ta tenglamalar sistemasini qaraymiz:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_m \end{cases} \tag{3.1}$$

Bu yerda, $x_1, x_2, ..., x_n$ – noma’lumlar, $a_{11}, a_{12}, ..., a_{mn}$ -koeffitsientlar, $b_1, b_2, ..., b_n$ -ozod hadlar.

(3.1) tenglamalar sistemasining *yechimi* deb, shunday n ta $\left(x_1{}^o, x_2{}^o, ..., x_n{}^o\right)$ sonlar to‘plamiga aytiladiki, bu sonlar (3.1) sistemaning barcha tenglamalarini to‘g‘ri tenglikka aylantiradi.

Agar (3.1) sistema yechimga ega bo‘lsa, u *birgalikdagi sistema* deyiladi. Agar bu yechim yagona bo‘lsa, sistema *aniq sistema* deyiladi. Agar (3.1) sistema cheksiz ko‘p yechimga ega bo‘lsa, u *aniqmas* sistema, agar tenglamalar sistemasi umuman yechimga ega bo‘lmasa, u *birgalikda bo‘lmagan sistema* deyiladi.

(3.1) tenglamalar sistemasi uchun quyidagi matritsalarni tuzamiz:

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{bmatrix} \tag{3.2}$$

A matritsa (3.1) sistemaning *asosiy matritsasi* deyiladi. B matritsa *kengaytirilgan matritsa* deyiladi.

Bu matritsalarning ranglar $rangA \le rangB$. munosabat bilan bog‘langan.

Agar A matritsaning rangi n noma’lumlar sonidan kichik bo‘lsa, u holda bu tenglamalar sistemasida $n-k$ ta o‘zgaruvchi chiziqli erkli bo‘lib, k ta o‘zgaruvchi chiziqli bog‘liq o‘zgaruvchilar bo‘ladi. Bu holda (3.1) tenglamalar sistemasida k ta tenglama qoldiriladi. Qolgan tenglamalar bu tenglamalarning chiziqli kombinatsiyasidan iborat bo‘ladi. Qoldirilgan tenglamalarda $n-k$ ta o‘zgaruvchini tenglamalarning o‘ng tomoniga o‘tkaziladi. Bu o‘zgaruvchilar chiziqli erkli o‘zgaruvchilar deyiladi. Tenglamalarni yechishda chiziqli erkli o‘zgaruvchilarga qiymatlar berilib, qolgan k ta o‘zgaruvchilarning ularga mos qiymatlari topiladi.

1. Kroniker – Kopelli teoremasi

1-teorema (*Kroniker – Kopelli*). *Chiziqli algebraik tenglamalar sistemasi birgalikda bo‘lishi uchun uning asosiy matritsasi bilan kengaytirilgan matritsasining rangi teng bo‘lishi zarur va yetarli, ya’ni* $rangA = rangB$.

Shunday qilib: $rangA \neq rangB$ bo‘lsa, tenglamalar sistemasi birgalikda emas;

$rangA = rangB = r = n$ bo‘lsa, tenglamalar sistemasi yagona yechimga ega;

$rangA = rang\ B = r < n$ bo‘lsa, tenglamalar sistemasi cheksiz ko‘p yechimga ega.

1.3.3 Bir jinsli chiziqli tenglamalar sistemasi

Agar chiziqli algebraik tenglamalar sistemasining barcha ozod hadlari nolga teng bo‘lsa, bunday sistema *bir jinsli chiziqli tenglamalar sistemasi* deyiladi.

Ushbu
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = 0 \end{cases} \tag{3.3}$$

tenglamalar sistemasi bir jinsli tenglamalar sistemasi.

Bu yerda $b_1 = b_2 = \ldots = b_m = 0$ bo‘lib A va B matritsalar ranglari teng,

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & 0 \end{bmatrix}$$

ya’ni *r(A)=r(B).*

Kroneker-Kapelli teoremasiga ko‘ra, bir jinsli chiziqli tenglamalar sistemasi hamma vaqt birgalikda bo‘ladi. (3.3) tenglamalar sistemasi doim nollardan iborat *trivial yechim* deb ataladigan yechimga ega:

$$x_1 = x_2 = \ldots = x_n = 0 \tag{3.4}$$

2-teorema. *Bir jinsli chiziqli tenglamalar sistemasining determinanti nolga teng bo‘lganda, va faqat shu holdagina bu sistema noldan farqli yechimlarga ega bo‘ladi.*

3-teorema. *(3.3) tenglamalar sistemasi noldan farqli yechimga ega bo'lishi uchun A matritsaning rangi noma'lumlar sonidan kichik, ya'ni* $r(A) < n$ *bo'lishi zarur va yetarli.*

Bir jinsli chiziqli tenglamalar sistemasining yechimlarining har qanday chiziqli kombinatsiyasi yana shu sistemaning yechimi bo'ladi.

$r(A)=k<n$ bo'lsa, u holda (3.3) sistemaning fundamental yechimlar sistemasi $n-k$ ta yechimdan iborat bo'ladi. Fundamental yechimlar sistemasini aniqlash uchun bazis noma'lumlarni aniqlaymiz. Ularni $x_1, x_2, \dots, x_k$ deb belgilaymiz. Bu noma'lumlarni $x_{k+1}, x_{k+2}, \dots, x_n$ chiziqli erkli noma'lumlar orqali ifodalab olinadi. Bu *n-k* ta noma'lumga ixtiyoriy qiymatlar berib, $x_1, x_2, \dots, x_k$ o'zgaruvchilarning mos aniq qiymatlarini topamiz. Bu topilgan yechimlar (3.3) ning fundamental yechimlar sistemasi bo'ladi. Ko'pincha normallangan fundamental yechimlar sistemasi olinadi.

2-misol

Fundamental va umumiy yechimlar sistemasi topilsin.

$$\begin{cases} x_1+2x_2+4x_3-3x_4=0 \\ 3x_1+5x_2+6x_3-4x_4=0 \\ 4x_1+5x_2-2x_3+3x_4=0 \\ 3x_1+8x_2+24x_3+19x_4=0 \end{cases}$$

► $A=\begin{bmatrix} 1 & 2 & 4 & -3 \\ 3 & 5 & 6 & -4 \\ 4 & 5 & -2 & 3 \\ 3 & 8 & 24 & 19 \end{bmatrix}$, $\quad M_2=\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}=5-6=-1\neq 0$

$$\begin{vmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \\ 4 & 5 & -2 \end{vmatrix}=0, \quad \begin{vmatrix} 1 & 2 & -3 \\ 3 & 5 & -4 \\ 4 & 5 & 3 \end{vmatrix}=0, \quad \begin{vmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \\ 3 & 8 & 24 \end{vmatrix}=0, \quad \begin{vmatrix} 1 & 2 & -3 \\ 3 & 5 & -4 \\ 3 & 8 & 19 \end{vmatrix}=0.$$

$$rang A=2, \begin{cases} x_1+2x_2+4x_3-3x_4=0 \\ 3x_1+5x_2+6x_3-4x_4=0 \end{cases}, \quad \begin{cases} x_1=8x_3-7x_4 \\ x_2=-6x_3+5x_4 \end{cases}.$$

$x_3=1,\ x_4=0$ va $x_3=0,\ x_4=1$ deb olib, $(8;-6;1;0)$ va $(-7;5;0;1)$ fundamental yechimlarni hosil qilamiz. Umumiy yechim: $\{8a-7b;-6a+5b;a;b\}$ ◄

Natija. *Agar bir jinsli tenglamalar sistemasining tenglamalari soni noma'lumlar sonidan kichik bo'lsa, bu sistema nolmas yechimga ega bo'ladi va bu yechimlar cheksiz ko'p bo'ladi.*

Agar bir jinsli tenglamalar sistemasining rangi $r < n$ bo‘lsa sistemadagi shu rangni tashkil qiluvchi minorlar turgan satrdagi tenglamalarni ajratamiz, ular qolgan $n-r$ dona tenglamalarning chiziqli kombinatsiyalardan iborat bo‘ladi.

3-misol

Fundamental va umumiy yechimlar sistemasi topilsin

$$\begin{cases} 2x_1 - 4x_2 + 5x_3 + 3x_4 = 0 \\ x_1 - 6x_2 + 4x_3 + 2x_4 = 0 \\ 4x_1 - 8x_2 + 17x_3 + 11x_4 = 0 \end{cases}$$

► $A = \begin{bmatrix} 2 & -4 & 5 & 3 \\ 3 & -6 & 4 & 2 \\ 4 & -8 & 17 & 11 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 5 & 3 \\ 0 & 0 & -7 & -5 \\ 0 & 0 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 5 & 3 \\ 0 & 0 & -7 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

$rang(A) = 2$.

$$\begin{cases} 2x_1 - 4x_2 + 5x_3 + 3x_4 = 0 \\ -7x_3 - 5x_4 = 0 \end{cases} \qquad \begin{cases} x_1 = 2x_2 - 4{,}6x_3 \\ x_4 = -1{,}4x_3 \end{cases}$$

Javob: $\{2a - 4{,}6b;\ a;\ b; -1.4b\}$.◄

Auditoriya topshiriqlari

1. Berilgan A matritsaning rangini ta’rif yordamida toping.

a) $A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & -4 & 4 \\ 3 & 6 & -6 \end{bmatrix}$; b) $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 3 & 4 \\ 4 & 3 & -8 \end{bmatrix}$; d) $A = \begin{bmatrix} -2 & 0 & -2 \\ 2 & -3 & 4 \\ -4 & 3 & -8 \end{bmatrix}$.

2. Berilgan A matritsaning rangini o‘rab turuvchi minorlar usulida yeching.

$a)\quad A = \begin{bmatrix} 3 & -1 & 2 & 1 \\ 2 & -1 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ $\qquad b)\quad A = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 2 & -1 & 1 & 5 \\ 0 & 1 & 1 & 3 \end{bmatrix}$

$c)\quad A = \begin{bmatrix} 1 & -2 & 2 & 1 \\ 2 & -1 & 4 & 2 \\ 4 & 0 & 1 & 1 \\ 5 & -2 & 3 & 2 \end{bmatrix}$ $\qquad d)\quad A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 1 & 4 & 2 \\ -1 & 4 & -3 & 1 \\ 2 & -2 & 5 & -2 \end{bmatrix}$

3. Berilgan A matritsaning rangini elementar almashtirishlar usulida yeching.

$a)\quad A = \begin{bmatrix} 1 & 3 & -2 & 0 & -1 & 4 \\ 2 & 1 & 3 & 2 & 5 & 3 \\ -1 & 4 & 1 & 3 & 0 & 3 \\ 5 & 2 & 4 & 1 & 9 & 7 \\ 4 & -1 & 3 & -2 & 7 & 3 \end{bmatrix}$ $\qquad b)\quad A = \begin{bmatrix} 1 & 5 & 2 & 1 & -2 & 2 \\ -1 & 1 & 3 & 2 & -2 & 5 \\ 2 & 4 & -1 & -3 & 0 & -3 \\ 3 & 9 & 1 & -2 & -2 & -1 \\ 5 & 1 & -5 & -8 & 4 & -16 \end{bmatrix}$

4. Sistema birgalikda bo‘ladimi?

$$a)\ \begin{cases} x_1 - 2x_2 + 5x_3 = 0 \\ 2x_1 - 5x_2 - 2x_4 = -2 \\ 3x_1 + 2x_2 + 6x_3 = 16 \end{cases} \qquad b)\ \begin{cases} 2x_1 + 3x_2 - 7x_3 = 2 \\ 3x_1 - x_2 + x_4 = 9 \\ 4x_1 - 5x_2 + 9x_3 = 14 \end{cases}$$

5. Bir jinsli chiziqli algebraik tenglamalar sistemasini yeching.

$$a)\ \begin{cases} x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 - 2x_2 + 5x_4 = 0 \\ 2x_1 - 2x_2 + 3x_3 = 0 \end{cases} \qquad b)\ \begin{cases} 2x_1 - 4x_2 + 5x_3 = 0 \\ x_1 + 2x_2 - 3x_4 = 0 \\ 3x_1 - 2x_2 + 2x_3 = 0 \end{cases}$$

6. Fundamental va umumiy yechimlar sistemasi topilsin.

$$a)\ \begin{cases} 4x_1 - 2x_2 - x_3 + 3x_4 = 0 \\ 2x_1 + 8x_2 + 3x_3 - 5x_4 = 0 \\ x_1 - 5x_2 - 2x_3 + 4x_4 = 0 \\ 3x_1 + 3x_2 + x_3 - x_4 = 0 \end{cases} \qquad b)\ \begin{cases} x_1 - 5x_2 + 2x_3 + 4x_4 = 0 \\ 2x_1 - 3x_2 - 3x_3 + x_4 = 0 \\ x_1 + 2x_2 - 5x_3 - 3x_4 = 0 \\ 3x_1 - 8x_2 - x_3 + 5x_4 = 0 \end{cases}$$

Mustaqil yechish uchun testlar

1. Matitsa rangini toping.

$$A = \begin{bmatrix} -1 & 3 & 2 & 4 \\ 1 & 5 & 6 & 4 \\ 3 & 0 & 3 & -3 \end{bmatrix}$$

$A)\ rang(A) = 3 \quad B)\ rang(A) = 2 \quad C)\ rang(A) = 1 \quad D)\ rang(A) = 4$

2. Matritsa rangi bu -

A) Matritsaning o‘lchami B) matritsaning determinanti.

C) noldan farqli eng katta minorining qiymati D) noldan farqli minorlarining eng katta tartibi

3. Matritsaning ko‘rsatilgan minorini o‘rab turuvchi minori noto‘g‘ri berilgan variantni aniqlang:

$$A = \begin{bmatrix} 1 & 3 & 2 & -4 \\ 1 & 5 & 6 & 4 \\ 3 & 0 & 3 & -3 \\ -2 & 4 & 5 & -1 \end{bmatrix}; \quad M_2 = \begin{vmatrix} 1 & 6 \\ 3 & 3 \end{vmatrix}$$

$$A)\ \begin{vmatrix} 1 & 3 & 2 \\ 1 & 5 & 6 \\ 3 & 0 & 3 \end{vmatrix}, \quad B)\ \begin{vmatrix} 1 & 6 & 4 \\ 3 & 3 & -3 \\ -2 & 5 & -1 \end{vmatrix}, \quad C)\ \begin{vmatrix} 1 & 2 & -4 \\ 1 & 6 & 4 \\ 3 & 3 & -1 \end{vmatrix}, \quad D)\ \begin{vmatrix} 1 & 5 & 6 \\ 3 & 0 & 3 \\ -2 & 4 & 5 \end{vmatrix}$$

4. O'lchami 4×4 bo'lgan matritsaning nechta 2-tartibli va nechta 3-tartibli minorlari mavjud?

A) 16 va 9; B) 25 va 16; C) 36 va 9; D) 36 va 16.

5. Agar n ta noma'lumli chiziqli algebraik tenglamalar sistemasi aniqmas sistema bo'lsa. uning asosiy A va kengaytirilgan B matritsalari ranglari qanday bog'langan bo'ladi?

$A)$ $rang(A)=rang(B)<n$ $B)$ $rang(A)\neq rang(B)$

C) $rang(A)=rang(B)=n$ $D)$ $rang(A)<n$

6. Agar 5 ta noma'lumli chiziqli bir jinsli algebraik tenglamalar sistemasi asosiy matritsasi rangi $r=3$ bo'lsa, uning fundamental yechimlari soni nechta bo'ladi?

A) 3ta; B) 2ta; D) 5ta; E) 4ta.

1.4 Chiziqli algebraik tenglamalar sistemasini yechish usullari

1.4.1 Chiziqli tenglamalar sistemasini yechishning Kramer usuli.

Determinantlarni chiziqli tenglamalar sistemasini yechishga tatbiqi bo'lgan *Kramer(determinant) usuli* bilan tanishamiz. Aytaylik bizga n ta no'malumli n ta chiziqli tenglamalar sistemasi berilgan bo'lsin.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots.. + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots.. + a_{nn}x_n = b_n \end{cases} \qquad (4.1)$$

Bu yerda $x_1,x_2,...,x_n$ -noma'lumlar, $a_{11},a_{12},...,a_{nn}$ -koeffitsientlar, $b_1,b_2,...,b_n$ -ozod sonlar.

Teorema. *Agar (4.1)-tenglamalar sistemasining asosiy determinanti(* $\Delta\neq0$ *) noldan farqli bo'lsa, u holda sistema yagona yechimga ega bo'ladi va u quyidagi formulalardan topiladi .*

$$\Delta\neq0\ ,\ x_1=\frac{\Delta_{x_1}}{\Delta},\ x_2=\frac{\Delta_{x_2}}{\Delta},\ ...,x_n=\frac{\Delta_{x_n}}{\Delta} \qquad (4.2)$$

Bu *Kramer* formulasidan iborat. Bu yerda $\Delta\neq0$ ga bosh determinant, $\Delta_{x_1},\Delta_{x_2},\Delta_{x_3},...,\Delta_{x_n}$ larga yordamchi determinantlar deyiladi. Soddalik uchun uch noma'lumli, uchta chiziqli tenglamalar sistemasini qaraymiz.

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases} \qquad (4.3)$$

uch noma'lumli uchta chiziqli tenglamalar sistemasini yechishda dastlab bosh(asosiy) determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad (4.4)$$

topiladi. $\Delta \neq 0$ bo'lsin. Undan so'ng yordamchi determinantlar hisoblanadi(bunda bosh determinantning ustun elementlari mos ravsihda ozod hadlar bilan almashtiriladi):

$$\Delta_x = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad \Delta_z = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix} \qquad (4.5)$$

Noma'lumlar quyidagi formulalar yordamida hisoblanadi:

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}, \quad z = \frac{\Delta_z}{\Delta} \qquad (4.6)$$

1-misol

Ushbu sistemani Kramer usulida yeching:

$$\begin{cases} x + 5y - z = 3, \\ 2x + 4y - 3z = 2, \\ 3x - y - 3z = -7 \end{cases}.$$

►Quyidagi determinantlarni tuzamiz va hisoblaymiz:

$$\Delta = \begin{vmatrix} 1 & 5 & -1 \\ 2 & 4 & -3 \\ 3 & -1 & -3 \end{vmatrix} = -16; \qquad \Delta_x = \begin{vmatrix} 3 & 5 & -1 \\ 2 & 4 & -3 \\ -7 & -1 & -3 \end{vmatrix} = 64;$$

$$\Delta_y = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 2 & -3 \\ 3 & -7 & -3 \end{vmatrix} = -16; \qquad \Delta_z = \begin{vmatrix} 1 & 5 & 3 \\ 2 & 4 & 2 \\ 3 & -1 & -7 \end{vmatrix} = 32.$$

Bundan, $x = \dfrac{64}{-16} = -4, \quad y = \dfrac{-16}{-16} = 1, \quad z = \dfrac{32}{-16} = -2.$ ◄

Agar bosh determinant nolga teng bo''lsa, tenglamalar sistemasi yechimga ega bo'lmaydi yoki cheksiz ko'p yechimga ega bo'ladi. Ya'ni

1) $\Delta = 0$ bo'lib, Δ_x, Δ_y, Δ_z lardan kamida bittasi nolga teng bo'lmasa, (4.3) tengamalar sistemasi yechimga ega bo'lmaydi,

2) $\Delta = 0$ bo'lib, $\Delta_x = 0$, $\Delta_y = 0$, $\Delta_z = 0$ bo'lsa, sistema cheksiz ko'p yechimga ega bo'ladi.

2-misol

Ushbu sistemani Kramer usulida yeching:

$$\begin{cases} x+2y-3z=7 \\ 2x+\ y-2z=9 \\ 3x \qquad -z=10 \end{cases}$$

► Bosh determinantini hisoblaymiz:

$$\Delta=\begin{vmatrix} 1 & 2 & -3 \\ 2 & 1 & -2 \\ 3 & 0 & -1 \end{vmatrix}=-1-12+0+9-0+4=0\,.$$

Yordamchi determinantlarni hisoblaymiz:

$$\Delta_x=\begin{vmatrix} 7 & 2 & -3 \\ 9 & 1 & -2 \\ 10 & 0 & -1 \end{vmatrix}=-7-40+0+30-0+18=1\,.$$

$\Delta=0$ bo‘lib, $\Delta_x=1\neq 0$ bo‘lgani uchun berilgan tenglamalar sistemasi yechimga ega emas.◄

3-misol

Ushbu sistemani Kramer usulida yeching:

$$\begin{cases} x-2y+z=5 \\ 2x \qquad -z=3 \\ 3x+2y-3z=1 \end{cases}$$

►Quyidagi determinantlarni tuzamiz va hisoblaymiz:

$$\Delta=\begin{vmatrix} 1 & -2 & 1 \\ 2 & 0 & -1 \\ 3 & 2 & -3 \end{vmatrix}=0+6+4-0+2-12=0\,,$$

$$\Delta_x=\begin{vmatrix} 5 & -2 & 1 \\ 3 & 0 & -1 \\ 1 & 2 & -3 \end{vmatrix}=0+2+6-0+10-18=0\,,$$

$$\Delta_y=\begin{vmatrix} 1 & 5 & 1 \\ 2 & 3 & -1 \\ 3 & 1 & -3 \end{vmatrix}=-9-15+2-9+1+30=0\,,$$

$$\Delta_z=\begin{vmatrix} 1 & -2 & 5 \\ 2 & 0 & 3 \\ 3 & 2 & 1 \end{vmatrix}=0-18+20-0-6+4=0\,.$$

$\Delta=0$ bo‘lib, $\Delta_x=0$, $\Delta_y=0$, $\Delta_z=0$ bo‘lgani uchun sistema cheksiz ko‘p yechimga ega bo‘ladi.

Bu holda 2 ta tenglamani qoldirib, erkli noma’lum, masalan, z ni tenlikning o‘ng tomoniga o‘tkazamiz.

$$\begin{cases} x-2y+z=5 \\ 2x \quad\quad -z=3 \end{cases} \qquad \begin{cases} x-2y=5-z \\ 2x \quad\quad =3+z \end{cases}$$

Hosil bo‘lgan ikki noma’lumli tenglamalar sistemasini yana Kramer usulida yechamiz.

$$\Delta=\begin{vmatrix} 1 & -2 \\ 2 & 0 \end{vmatrix}=4,\ \Delta_x=\begin{vmatrix} 5-z & -2 \\ 3+z & 0 \end{vmatrix}=6+2z,\ \Delta_y=\begin{vmatrix} 1 & 5-z \\ 2 & 3+z \end{vmatrix}=-7+3z.$$

Demak, tenglamaning yechimi: $\left\{\dfrac{z+3}{2}; \dfrac{3z-7}{4}; z\right\}$.◄

1.4.2 Chiziqli algebraik tenglamalar sistemasini yechishning matrisa usuli.

Aytaylik bizga n ta no‘malumli n ta chiziqli (4.1) tenglamalar sistemasi berilgan bo‘lsin. Ushbu belgilashlarni kiritamiz:

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}; \qquad X=\begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}; \qquad B=\begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix} \qquad (4.7)$$

U holda (4.1) sistemani matrisalarni ko‘paytirish qoidasidan foydalanib, ushbu ekvivalent shaklda yozish mumkin:

$$A\cdot X=B \qquad (4.8)$$

Bu yerda A -noma’lumlar oldidagi koeffisientlardan tuzilgan matrisa, B -ozod hadlardan tuzilgan ustun matrisa, X -noma’lumlardan tuzilgan ustun matrisa

Agar A matrisa xosmas, ya’ni $\det A\neq 0$ bo‘lsa, u holda uning uchun A^{-1} teskari matrisa mavjud. (4.8) matrisali tenglamaning ikkala qismini A^{-1} ga chapdan ko‘paytirib, quyidagini hosil qilamiz:

$$A^{-1}\cdot(A\cdot X)=A^{-1}\cdot B$$

yoki

$$\left(A^{-1}\cdot A\right)\cdot X=A^{-1}\cdot B.$$

$A^{-1}\cdot A=E,\quad E\cdot X=X$ ekanligini hisobga olib,

$$X=A^{-1}\cdot B \qquad (4.9)$$

ni topamiz. (4.9) formula A matrisa xosmas bo‘lganda n no‘malumli n t ta chiziqli tenglamalar *sistemasi yechimining matritsali yozuvidan* iborat bo‘ladi.

4-misol

Ushbu sistemani yeching:

$$\begin{cases} x_1 - 2x_2 + x_3 = 5 \\ 2x_1 - x_3 = 0 \\ -2x_1 + x_2 + x_3 = -1. \end{cases}$$

►Bu yerda

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 0 & -1 \\ -2 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}, \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$detA = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 0 & -1 \\ -2 & 1 & 1 \end{vmatrix} = -4 + 2 + 1 + 4 = 3 \neq 0,$$

$$A_{11} = 1, \; A_{12} = 0, \; A_{13} = 2, \; A_{21} = 3, \; A_{22} = 3, \; A_{23} = 3, \; A_{31} = 2, \; A_{32} = 3, \; A_{33} = 4$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 1 & \frac{2}{3} \\ 0 & 1 & 1 \\ \frac{2}{3} & 1 & \frac{4}{3} \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$x_1 = \frac{1}{3} \cdot 5 + 1 \cdot 0 + \frac{2}{3} \cdot (-1) = \frac{5}{3} - \frac{2}{3} = \frac{3}{3} = 1;$$

$$x_2 = 0 \cdot 5 + 1 \cdot 0 + 1 \cdot (-1) = 0 - 1 = -1;$$

$$x_3 = \frac{2}{3} \cdot 5 + 1 \cdot 0 + \frac{4}{3} \cdot (-1) = \frac{10}{3} - \frac{4}{3} = \frac{6}{3} = 2.$$

Bundan $x_1 = 1, \; x_2 = -1, \; x_3 = 2.$ ◄

1.4.2 Chiziqli algebraik tenglamalar sistemasini yechishning Gauss usuli.

Bizga n ta noma'lumli n ta chiziqli (4.1) tenglamalar sistemasi berilgan bo‘lsin. Uning asosiy matritsasi A ning rangi $rang(A) = r \leq n$ bo‘lsa, kengaytirilgan matritsasi B ni har doim, elementar almashtirishlar yordamida, quyidagi ekvivalent matritsaga almashtirish mumkin.

$$\begin{bmatrix} 1 & \bar{a}_{12} & \dots & \bar{a}_{1r} & \bar{a}_{1\,r+1} & \dots & \bar{a}_{1n} & \bar{b}_1 \\ 0 & 1 & \dots & \bar{a}_{2r} & \bar{a}_{2\,r+1} & \dots & \bar{a}_{2n} & \bar{b}_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \bar{a}_{r\,r+1} & \dots & \bar{a}_{rn} & \bar{b}_r \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 & \bar{b}_{r+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 & \bar{b}_n \end{bmatrix} \qquad (4.10)$$

Agar bu matritsada $\bar{b}_{r+1}.\bar{b}_{r+2},\dots,\bar{b}_n$ lardan kamida bittasi noldan farqli bo‘lsa, (4.1) sistema yechimga ega emas, chunki $rang(A) \neq rang(B)$ bo‘ladi. Agar $\bar{b}_{r+1} = \bar{b}_{r+2} = \dots = \bar{b}_n = 0$ bo‘lsa, berilgan (4.1) chiziqli algebraik tenglamalar sistemasi birgalikda bo‘ladi. Bu holda (4.10) matritsaning bazis satrlariga mos tenglamalarni tuzamiz.

$$\begin{cases} x_1 + \bar{a}_{12}x_2 + \dots + \bar{a}_{1r}x_r + \bar{a}_{1r+1}x_{r+1} + \dots + \bar{a}_{1n}x_n = \bar{b}_1 \\ \qquad x_2 + \dots + \bar{a}_{2r}x_r + \bar{a}_{2r+1}x_{r+1} + \dots + \bar{a}_{2n}x_n = \bar{b}_2 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \dots \quad \dots \quad \dots \quad x_r + \bar{a}_{r\,r+1}x_{r+1} + \dots + \bar{a}_{rn}x_n = \bar{b}_r \end{cases} \qquad (4.11)$$

Hosil bo‘lgan (4.11) sistemaning yechimlari berilgan (4.1) chiziqli algebraik tenglamalar sistemasining ham yechimlaridir. (4.11) da $x_r, x_{r-1},\dots,x_2, x_1$ bazis noma’lumlarni $x_{r+1}, x_{r+2},\dots,x_n$ erkli nomalumlar orqali, oxirgi tenglamadan boshlab ketma-ket aniqlanadi. Agar $r = n$ bo‘lsa, chiziqli algebraik tenglamalar sistemasining yechimi yagona bo‘ladi.

Gauss usuli n ta noma’lumli m ta chiziqli tenglamalar sistemasi bo‘lgan holda ham o‘rinli bo‘ladi.

3-misol

Ushbu sistemani Gauss usulida yeching:

$$\begin{cases} x + y + z = 6 \\ 2x + 2y - 3z = -3 \\ 3x - y + 2z = 7 \end{cases}$$

► Berilgan tenglamalar sistemasining kengaytirilgan matritsasini tuzamiz va ekvivalent almashtirishlar yordamida quyidagi ekvivalent matritsani hosil qilamiz.

$$B = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 2 & -3 & -3 \\ 3 & -1 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 0 & -5 & -15 \\ 0 & -4 & -1 & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 4 & 1 & 11 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Bundan sistema birgalida ekani kelib chiqadi, chunki $rang(A) = rang(B) = 3$.

$$\begin{cases} x + y + z = 6 \\ \quad 4y + z = 11 \\ \qquad z = 3 \end{cases}$$

sistemaga ega bo‘ldik. Oxirgi tenglamadan boshlab, $z = 3$ ni ikkinchi tenglamaga qo‘yib, y ni topamiz:

$$4y+3=11,\; y=2.$$

Endi $y=2$ va $z=3$ ni 1- tenglamaga qo‘yib, x ni topamiz:

$$x+2+3=6,\; x=1.$$

Demak, $x=1, y=2, z=3$.◄

Auditoriya topshiriqlari

1. Berilgan chiziqli algebraik tenglamalar sistemasini Kramer usulida yeching

$$a)\ \begin{cases} x_1+2x_2+3x_3=7 \\ 2x_1-5x_2+x_3=4 \\ 3x_1+3x_2-5x_3=-7 \end{cases} \qquad b)\ \begin{cases} 3x_1+2x_2+5x_3=0 \\ 5x_1+x_3=4 \\ 2x_1+3x_2=5 \end{cases} \qquad d)\ \begin{cases} x_1-2x_2+4x_3=6 \\ 2x_1+5x_2-6x_3=7 \\ 3x_1+3x_2-2x_3=8 \end{cases}$$

2. Berilgan chiziqli algebraik tenglamalar sistemasini matritsa usulida yeching

$$a)\ \begin{cases} x_1+2x_2-x_3=-2 \\ 2x_1-x_2=-1 \\ x_2+x_3=-2 \end{cases} \qquad b)\ \begin{cases} x_1+2x_2+x_3=1 \\ 4x_1+2x_2-x_3=0 \\ x_2-x_3=-3 \end{cases}$$

3. Berilgan chiziqli algebraik tenglamalar sistemasini Gauss usulida yeching.

$$a)\ \begin{cases} 3x_1+2x_2+x_3=3 \\ 2x_1+3x_2-2x_3=-1 \\ x_1+x_2-5x_3=6 \end{cases} \qquad b)\ \begin{cases} 4x_1+2x_2-3x_3+2x_4=3 \\ 2x_1+3x_2-2x_3+3x_4=2 \\ 3x_1+2x_2-3x_3+4x_4=1 \end{cases}$$

$$c)\ \begin{cases} 2x_1+5x_2-8x_3=8 \\ 2x_1+3x_2-5x_3=7 \\ x_1+8x_2-7x_3=12 \\ 4x_1+3x_2-9x_3=9 \end{cases} \qquad d)\ \begin{cases} 2x_1+x_2+3x_3+2x_4=-3 \\ x_1+x_2+5x_3+2x_4=1 \\ 3x_1+3x_2+9x_3+5x_4=-2 \\ 2x_1+3x_2+11x_3+5x_4=2 \end{cases}$$

Shaxsiy uy topshiriqlari

1

Berilgan Δ determinant uchun a_{i2}, a_{3j} elementlarning minorlari va algebraik to‘ldiruvchilarini toping. Δ determinantni:

a) *i*-satr elementlari bo‘yicha yoyib;

b) *j*-ustun elementlari bo‘yicha yoyib;

d) *i*-satr elementlarini nollarga aylantirib hisoblang.

1.1. $\begin{vmatrix} 1 & 1 & -2 & 0 \\ 3 & 6 & -2 & 5 \\ 1 & 0 & 6 & 4 \\ 2 & 3 & 5 & -14 \end{vmatrix}$ $i=4,\ j=1$

1.2. $\begin{vmatrix} 2 & 0 & -1 & 3 \\ 6 & 3 & -9 & 0 \\ 0 & 2 & -1 & 3 \\ 4 & 2 & 0 & 6 \end{vmatrix}$ $i=1,\ j=3$

1.3. $\begin{vmatrix} 2 & 7 & 2 & 1 \\ 1 & 1 & -1 & 0 \\ 3 & 4 & 0 & 2 \\ 0 & 5 & -1 & -3 \end{vmatrix}$ $i=4,\ j=1$

1.4. $\begin{vmatrix} 4 & -5 & -1 & -5 \\ -3 & 2 & 8 & -2 \\ 5 & 3 & 1 & 3 \\ -2 & 4 & -6 & 8 \end{vmatrix}$ $i=1,\ j=3$

1.5. $\begin{vmatrix} 3 & 5 & 3 & 2 \\ 2 & 4 & 1 & 0 \\ 1 & -2 & 2 & 1 \\ 5 & 1 & -2 & 4 \end{vmatrix}$ $i=2,\ j=4$

1.6. $\begin{vmatrix} 3 & 2 & 0 & -5 \\ 4 & 3 & -5 & 0 \\ 1 & 0 & -2 & 3 \\ 0 & 1 & -3 & 4 \end{vmatrix}$ $i=1,\ j=2$

1.7. $\begin{vmatrix} 2 & -1 & 2 & 0 \\ 3 & 4 & 1 & 2 \\ 2 & -1 & 0 & 1 \\ 1 & 2 & 3 & -2 \end{vmatrix}$ $i=2,\ j=3$

1.8. $\begin{vmatrix} 3 & 2 & 0 & -2 \\ 1 & -1 & 2 & 3 \\ 4 & 5 & 1 & 0 \\ -1 & 2 & 3 & -3 \end{vmatrix}$ $i=3,\ j=1$

1.9. $\begin{vmatrix} 0 & -1 & 2 & 0 \\ 3 & 4 & 1 & 2 \\ 2 & -1 & 0 & 1 \\ 1 & 2 & 3 & -2 \end{vmatrix}$ $i=4,\ j=3$

1.10. $\begin{vmatrix} 0 & -2 & 1 & 7 \\ 4 & -8 & 2 & -3 \\ 10 & 1 & -5 & 4 \\ -8 & 3 & 2 & -1 \end{vmatrix}$ $i=4,\ j=2$

1.11. $\begin{vmatrix} 5 & -3 & 7 & -1 \\ 3 & 2 & 0 & 2 \\ 2 & 1 & 4 & -6 \\ 3 & -2 & 9 & 4 \end{vmatrix}$ $i=1,\ j=4$

1.12. $\begin{vmatrix} 4 & -1 & 1 & 5 \\ 0 & 2 & -2 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 1 & -2 \end{vmatrix}$ $i=2,\ j=4$

1.13. $$\begin{vmatrix} 1 & 8 & 2 & -3 \\ 3 & -2 & 0 & 4 \\ 5 & -3 & 7 & -1 \\ 3 & 2 & 0 & 2 \end{vmatrix}$$
$i=1,\ j=4$

1.14. $$\begin{vmatrix} 2 & -3 & 4 & 1 \\ 4 & -2 & 3 & 2 \\ 3 & 0 & 2 & 1 \\ 3 & -1 & 4 & 3 \end{vmatrix}$$
$i=2,\ \ j=4$

1.15. $$\begin{vmatrix} 3 & 1 & 2 & 3 \\ 4 & -1 & 2 & 4 \\ 1 & -1 & 1 & 1 \\ 4 & -1 & 2 & 5 \end{vmatrix}$$
$i=1,\ j=3$

1.16. $$\begin{vmatrix} 3 & 1 & 2 & 0 \\ 5 & 0 & -6 & 1 \\ -2 & 2 & 1 & 3 \\ -1 & 3 & 2 & 1 \end{vmatrix}$$
$i=3,\ \ j=2$

1.17. $$\begin{vmatrix} 1 & -1 & 0 & 3 \\ 3 & 2 & 1 & -1 \\ 1 & 2 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{vmatrix}$$
$i=3,\ j=1$

1.18. $$\begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$
$i=2,\ \ j=4$

1.19. $$\begin{vmatrix} 6 & 2 & -10 & 4 \\ -5 & -7 & -4 & 1 \\ 2 & 4 & -2 & -6 \\ 3 & 0 & -5 & 2 \end{vmatrix}$$
$i=2,\ j=3$

1.20. $$\begin{vmatrix} -1 & -2 & 4 & 1 \\ 2 & 3 & 0 & 6 \\ 2 & -2 & 1 & 4 \\ 3 & 1 & -2 & -1 \end{vmatrix}$$
$i=4,\ \ j=3$

1.21. $$\begin{vmatrix} 2 & 7 & 1 & 1 \\ 2 & 4 & 1 & 0 \\ 1 & -2 & 2 & 1 \\ 5 & 1 & -2 & 4 \end{vmatrix}$$
$i=4,\ j=2$

1.22. $$\begin{vmatrix} 1 & 2 & 0 & -5 \\ 0 & 1 & -5 & 5 \\ 1 & 0 & -2 & 3 \\ -1 & 1 & -3 & 4 \end{vmatrix}$$
$i=3,\ \ j=3$

1.23. $$\begin{vmatrix} 1 & 5 & -1 & 2 \\ 3 & 4 & 1 & 2 \\ 2 & -1 & 0 & 1 \\ 1 & 2 & 3 & -2 \end{vmatrix}$$
$i=2,\ j=4$

1.24. $$\begin{vmatrix} 2 & 4 & 3 & -5 \\ 1 & -1 & 2 & 3 \\ 4 & 5 & 1 & 0 \\ -1 & 2 & 3 & -3 \end{vmatrix}$$
$i=1,\ \ j=4$

1.25. $\begin{vmatrix} 0 & -1 & 2 & 0 \\ 3 & 4 & 1 & 2 \\ 2 & -1 & 0 & 1 \\ 1 & 2 & 3 & -2 \end{vmatrix}$ $i=3,\ j=1$

1.26. $\begin{vmatrix} 0 & -2 & 1 & 7 \\ 4 & -8 & 2 & -3 \\ 10 & 1 & -5 & 4 \\ -8 & 3 & 2 & -1 \end{vmatrix}$ $i=1,\ j=3$

1.27. $\begin{vmatrix} 5 & -3 & 7 & -1 \\ 3 & 2 & 0 & 2 \\ 2 & 1 & 4 & -6 \\ 3 & -2 & 9 & 4 \end{vmatrix}$ $i=2,\ j=4$

1.28. $\begin{vmatrix} 4 & -1 & 1 & 5 \\ 0 & 2 & -2 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 1 & -2 \end{vmatrix}$ $i=3,\ j=3$

1.29. $\begin{vmatrix} 1 & 8 & 2 & -3 \\ 3 & -2 & 0 & 4 \\ 5 & -3 & 7 & -1 \\ 3 & 2 & 0 & 2 \end{vmatrix}$ $i=3,\ j=1$

1.30. $\begin{vmatrix} 2 & -3 & 4 & 1 \\ 4 & -2 & 3 & 2 \\ 3 & 0 & 2 & 1 \\ 3 & -1 & 4 & 3 \end{vmatrix}$ $i=4,\ j=1$

2

Ikkita A va B matritsalar berilgan. Quyidagilarni toping:
a) $A\cdot B$; b) $B\cdot A$; d) A^{-1}.

2.1. $A=\begin{bmatrix} 2 & -1 & -3 \\ 8 & -7 & -6 \\ -3 & 4 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 2 & -1 & -2 \\ 3 & -5 & 4 \\ 1 & 2 & 1 \end{bmatrix}$.

2.2. $A=\begin{bmatrix} 3 & 5 & -6 \\ 2 & 4 & 3 \\ -3 & 1 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 8 & -5 \\ -3 & -1 & 0 \\ 4 & 5 & -3 \end{bmatrix}$.

2.3. $A=\begin{bmatrix} 2 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 6 & 0 \\ 2 & 4 & -6 \\ 1 & 2 & 1 \end{bmatrix}$.

2.4. $A=\begin{bmatrix} -6 & 1 & 11 \\ 9 & 2 & 5 \\ 0 & 4 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 7 \\ 1 & -3 & 2 \end{bmatrix}$.

2.5. $A=\begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 0 & -1 & 2 \\ 2 & 1 & 1 \\ 3 & 7 & 1 \end{bmatrix}$.

2.6. $A=\begin{bmatrix} 2 & 3 & 2 \\ 1 & 3 & -1 \\ 4 & 1 & 3 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 2 & -1 \\ 3 & 1 & 2 \\ 5 & 3 & 0 \end{bmatrix}$.

2.7. $A=\begin{bmatrix} 6 & 7 & 3 \\ 3 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 0 & 5 \\ 4 & -1 & -2 \\ 4 & 3 & 7 \end{bmatrix}$.

2.8. $A=\begin{bmatrix} 2 & 3 & 4 \\ 3 & -1 & -4 \\ -1 & 2 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 3 & 1 \\ 0 & 6 & 2 \\ 1 & 9 & 2 \end{bmatrix}$.

2.9. $A=\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 4 \\ 1 & 1 & 5 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 0 & 5 \\ 1 & 2 & 7 \\ 4 & 3 & 7 \end{bmatrix}$.

2.10. $A=\begin{bmatrix} 2 & 6 & 1 \\ 1 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 4 & -3 & 2 \\ -4 & 0 & 5 \\ 3 & 2 & -3 \end{bmatrix}$.

2.11. $A=\begin{bmatrix} 6 & 9 & 4 \\ -1 & -1 & 1 \\ 10 & 1 & 7 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 3 \\ 0 & 5 & 2 \end{bmatrix}$.

2.12. $A=\begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & 7 \\ 2 & 1 & 8 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 5 & 4 \\ -3 & 0 & 1 \\ 5 & 6 & -4 \end{bmatrix}$.

2.13. $A=\begin{bmatrix} 5 & 1 & -2 \\ 1 & 3 & -1 \\ 8 & 4 & -1 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 5 & 5 \\ 7 & 1 & 2 \\ 1 & 6 & 0 \end{bmatrix}$

2.14. $A=\begin{bmatrix} 2 & 2 & 5 \\ 3 & 3 & 6 \\ 4 & 3 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & -1 \end{bmatrix}$.

2.15. $A=\begin{bmatrix} 1 & -2 & 5 \\ 3 & 0 & 6 \\ 4 & 3 & 4 \end{bmatrix}$, $B=\begin{bmatrix} -1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & -1 \end{bmatrix}$.

2.16. $A=\begin{bmatrix} 5 & 4 & 2 \\ 1 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$, $B=\begin{bmatrix} 5 & 1 & -5 \\ 3 & -7 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

2.17. $A=\begin{bmatrix} 3 & 1 & 0 \\ 4 & 3 & 2 \\ 2 & 2 & -7 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 7 & 0 \\ 5 & 3 & 1 \\ 1 & -6 & 1 \end{bmatrix}$.

2.18. $A=\begin{bmatrix} 8 & -1 & -1 \\ 5 & -5 & -1 \\ 10 & 3 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 2 & 5 \\ 3 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.

2.19. $A=\begin{bmatrix} 3 & -7 & 2 \\ 1 & -8 & 3 \\ 4 & -2 & 3 \end{bmatrix}$, $B=\begin{bmatrix} 0 & 5 & -3 \\ 2 & 4 & 1 \\ 2 & 1 & -5 \end{bmatrix}$.

2.20. $A=\begin{bmatrix} 3 & -1 & 0 \\ 3 & 5 & 1 \\ 4 & -7 & 5 \end{bmatrix}$, $B=\begin{bmatrix} -1 & 0 & 2 \\ 1 & -8 & 4 \\ 3 & 0 & 2 \end{bmatrix}$.

2.21. $A=\begin{bmatrix} 0 & -3 & 6 \\ 2 & 1 & -2 \\ -3 & 4 & 2 \end{bmatrix}$, $B=\begin{bmatrix} -2 & 1 & 2 \\ 3 & -5 & 4 \\ 5 & -1 & 6 \end{bmatrix}$.

2.22. $A=\begin{bmatrix} 3 & 1 & -3 \\ 2 & 0 & 5 \\ -3 & 7 & -2 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 4 & 5 \\ -3 & 5 & 0 \\ 4 & -3 & 3 \end{bmatrix}$.

2.23. $A=\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ -2 & 1 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 6 & 0 \\ 2 & 4 & -6 \\ 1 & 2 & 1 \end{bmatrix}$.

2.24. $A=\begin{bmatrix} -6 & 1 & 11 \\ 9 & 2 & 5 \\ 0 & 4 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 7 \\ 1 & 1 & 4 \end{bmatrix}$.

2.25. $A=\begin{bmatrix} 4 & 1 & 2 \\ -1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 0 & 3 \\ 2 & 1 & 1 \\ 1 & 6 & 0 \end{bmatrix}$.

2.26. $A=\begin{bmatrix} 0 & -5 & -1 \\ 1 & 3 & -1 \\ 3 & -2 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 2 & -1 \\ 3 & 1 & 2 \\ 5 & 3 & 0 \end{bmatrix}$.

2.27. $A=\begin{bmatrix} 1 & 7 & 3 \\ -2 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 0 & 5 \\ 4 & -1 & -2 \\ 2 & 3 & 2 \end{bmatrix}$.

2.28. $A=\begin{bmatrix} 4 & 1 & 2 \\ -1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 6 & 0 \\ 2 & 4 & -6 \\ 1 & 2 & 1 \end{bmatrix}$.

2.29. $A=\begin{bmatrix} 1 & 5 & 6 \\ 3 & -1 & -4 \\ -1 & 2 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 2 & -6 & 1 \\ 0 & 6 & 2 \\ 1 & 3 & 0 \end{bmatrix}$.

2.30. $A=\begin{bmatrix} 2 & 5 & 0 \\ 1 & 3 & 2 \\ 0 & 4 & 3 \end{bmatrix}$, $B=\begin{bmatrix} 1 & -5 & 5 \\ 0 & -3 & 7 \\ 3 & 2 & -3 \end{bmatrix}$.

3. Chiziqli algebraik tenglamalar sistemasi birgalikda ekanligini tekshiring. Agar birgalikda bo‘lsa, uni

a) Kramer formulalari bo‘yicha;

b) matritsa usulida ;

d) Gauss usulida yeching.

3.1. $\begin{cases} 2x_1 + x_2 + 3x_3 = 7, \\ 2x_1 + 3x_2 + x_3 = 1, \\ 3x_1 + 2x_2 + x_3 = 6; \end{cases}$ 3.2. $\begin{cases} 2x_1 - x_2 + 2x_3 = 3, \\ x_1 + x_2 + 2x_3 = -4, \\ 4x_1 + x_2 + 4x_3 = -3; \end{cases}$

3.3. $\begin{cases} 3x_1 - x_2 + x_3 = 12, \\ x_1 + 2x_2 + 4x_3 = 6, \\ 5x_1 + x_2 + 2x_3 = 3; \end{cases}$ 3.4. $\begin{cases} 2x_1 - x_2 + 3x_3 = -4, \\ x_1 + 3x_2 - x_3 = 11, \\ x_1 - 2x_2 + 2x_3 = -7; \end{cases}$

3.5. $\begin{cases} 3x_1 - 2x_2 + 4x_3 = 12, \\ 3x_1 + 4x_2 - 2x_3 = 6, \\ 2x_1 - x_2 - x_3 = -9; \end{cases}$ 3.6. $\begin{cases} 8x_1 + 3x_2 - 6x_3 = -4, \\ x_1 + x_2 - x_3 = 2, \\ 4x_1 + x_2 - 3x_3 = -5; \end{cases}$

3.7. $\begin{cases} 4x_1 + x_2 - 3x_3 = 9, \\ x_1 - x_2 - x_3 = -2, \\ 8x_1 + 3x_2 - 6x_3 = 0; \end{cases}$ 3.8. $\begin{cases} 2x_1 + 3x_2 + 4x_3 = 33, \\ 7x_1 - 5x_2 = 24, \\ 4x_1 + x_3 = 39; \end{cases}$

3.9. $\begin{cases} 2x_1 + 3x_2 + 4x_3 = 12, \\ 7x_1 - 5x_2 + x_3 = -33, \\ 4x_1 + x_3 = -7; \end{cases}$ 3.10. $\begin{cases} x_1 + 4x_2 - x_3 = 6, \\ 5x_2 + 4x_3 = -20, \\ 3x_1 - 2x_2 + 5x_3 = -22; \end{cases}$

3.11. $\begin{cases} 3x_1 - 2x_2 + 4x_3 = 21, \\ 3x_1 + 4x_2 - 2x_3 = 9, \\ 2x_1 - x_2 - x_3 = 10; \end{cases}$ 3.12. $\begin{cases} 3x_1 - 2x_2 - 5x_3 = 5, \\ 2x_1 + 3x_2 - 4x_3 = 12, \\ x_1 - 2x_2 + 3x_3 = -1; \end{cases}$

3.13. $\begin{cases} 4x_1 + x_2 + 4x_3 = 19, \\ 2x_1 - x_2 + 2x_3 = 11, \\ x_1 + x_2 + 2x_3 = 8; \end{cases}$ 3.14. $\begin{cases} 2x_1 - x_2 + 2x_3 = 0, \\ 4x_1 + x_2 + 4x_3 = 6, \\ x_1 + x_2 + 2x_3 = 4; \end{cases}$

3.15. $\begin{cases} 2x_1 - x_2 + 2x_3 = 8, \\ x_1 + x_2 + 2x_3 = 11, \\ 4x_1 + x_2 + 4x_3 = 22; \end{cases}$ 3.16. $\begin{cases} 2x_1 - x_2 - 3x_3 = -9, \\ x_1 + 5x_2 + x_3 = 20, \\ 3x_1 + 4x_2 + 2x_3 = 15; \end{cases}$

3.17. $\begin{cases} 2x_1 - x_2 - 3x_3 = 0, \\ 3x_1 + 4x_2 + 2x_3 = 1, \\ x_1 + 5x_2 + x_3 = -3; \end{cases}$ 3.18. $\begin{cases} -3x_1 + 5x_2 + 6x_3 = -8, \\ 3x_1 + x_2 + x_3 = -4, \\ x_1 - 4x_2 - 2x_3 = -9; \end{cases}$

3.19. $\begin{cases} 3x_1 + x_2 + x_3 = -4, \\ -3x_1 + 5x_2 + 6x_3 = 36, \\ x_1 - 4x_2 - 2x_3 = -19; \end{cases}$ 3.20. $\begin{cases} 3x_1 - x_2 + x_3 = 11, \\ 5x_1 + x_2 + 2x_3 = 8, \\ x_1 + 2x_2 + 4x_3 = 16; \end{cases}$

3.21. $\begin{cases} x_1 - 3x_2 - 7x_3 = 0 \\ x_1 + 2x_2 + 4x_3 = 6, \\ 4x_1 - x_2 - 2x_3 = -3; \end{cases}$ 3.22. $\begin{cases} 3x_1 + 2x_2 + 2x_3 = 7, \\ x_1 + 3x_2 - x_3 = 11, \\ 3x_1 + 4x_2 = 15; \end{cases}$

3.23. $\begin{cases} x_1 - x_2 + 5x_3 = 21, \\ x_1 + 5x_2 - x_3 = 15, \\ 2x_1 - x_2 - x_3 = -9; \end{cases}$ 3.24. $\begin{cases} 6x_1 + x_2 - 4x_3 = -8, \\ x_1 + x_2 - x_3 = 2, \\ 4x_1 + x_2 - 3x_3 = -5; \end{cases}$

3.25. $\begin{cases} x_1 - x_2 - x_3 = -2, \\ 4x_1 + x_2 - 3x_3 = 9, \\ 4x_1 + 2x_2 - 3x_3 = -9; \end{cases}$ 3.26. $\begin{cases} 2x_1 - 3x_2 - 3x_3 = 6, \\ 5x_1 - 8x_2 - 4x_3 = -9, \\ 4x_1 + x_3 = 39; \end{cases}$

3.27. $\begin{cases} 2x_1 + 3x_2 + 4x_3 = 12, \\ 7x_1 - 8x_2 + 4x_3 = 38, \\ 4x_1 + x_3 = -7; \end{cases}$ 3.28. $\begin{cases} 7x_2 + 4x_3 = 20, \\ x_1 - x_2 - 5x_3 = 26, \\ x_1 - 3x_2 + 3x_3 = -14; \end{cases}$

3.29. $\begin{cases} x_1 - x_2 + 5x_3 = 11, \\ 3x_1 + 4x_2 - 2x_3 = 9, \\ 2x_1 + 5x_2 + 4x_3 = 10; \end{cases}$ 3.30. $\begin{cases} x_1 + x_2 + 7x_3 = 5, \\ x_1 - 9x_2 + 13x_3 = -15, \\ x_1 - 2x_2 + 3x_3 = -39. \end{cases}$

II BOB VEKTORLAR ALGEBRASI ELEMENTLARI

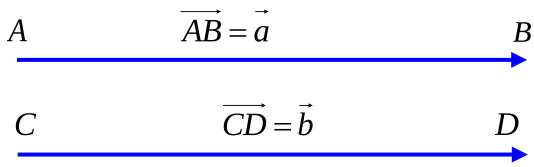
2.1 Vektorlar. Vektorlar ustida chiziqli amallar. Chiziqli bog'liq va chiziqli erkli vektorlar. Bazis

Vektorlar. Asosiy tushunchalar.Yo'nalgan kesma yoki nuqtalarning tartiblangan $\{A, B\}$ jufti *vektor* deyiladi; odatda birinchi nuqtani vektorning boshi, ikkinchi nuqtani esa uning oxiri (uchi) deyiladi(1-chizma) va $\overrightarrow{AB}$ kabi belgilanadi. Boshi va oxiri ko'rsatilmagan vektor lotin alifbosining kichik harflari bilan belgilanadi: $\vec{a}$, $\vec{b}$, …

1-chizma

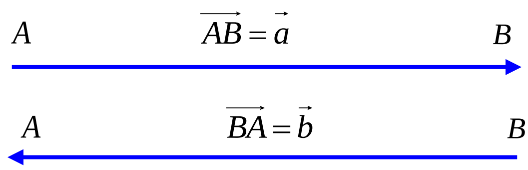
Vektorning *moduli* yoki *uzunligi* deb, vektorning boshi va oxiri orasidagi masofaga aytiladi. $|\overrightarrow{AB}|$ yoki $|\vec{a}|$ kabi belgilanadi. Bir to'g'ri chiziqda yoki parallel to'g'ri chiziqlarda yotuvchi vektorlar *kollinear vektorlar* deyiladi. Bir tekislikda yoki parallel tekisliklarda yotuvchi vektorlarga *komplanar vektorlar* deyiladi. Boshi va oxiri bir nuqtada bo'lgan vektor *nol vektor* deyiladi.

Uzunliklari teng, kollinear va yo'nalishlari bir xil bo'lgan ikki vektor *teng vektorlar* deb ataladi, boshqacha aytganda, agar $\vec{a}$ va $\vec{b}$ vektorlar uchun quyidagi uchta shart $\left(|\vec{a}| = |\vec{b}|,\ \vec{a} \parallel \vec{b},\ \vec{a} \uparrow\uparrow \vec{b}\right)$ bajarilsa, u holda $\vec{a}$ va $\vec{b}$ vektorlar teng deyiladi va $\vec{a} = \vec{b}$ deb yoziladi(2-chizma).



2-chizma

Uzunliklari teng, kollinear va yo'nalishlari har xil bo'lgan ikki vektorga *qarama-qarshi vektorlar* deyiladi, boshqacha aytganda, agar $\vec{a}$ va $\vec{b}$ vektorlar uchun quyidagi uchta shart $\left(|\vec{a}| = |\vec{b}|,\ \vec{a} \parallel \vec{b},\ \vec{a} \uparrow\downarrow \vec{b}\right)$ bajarilsa, u holda $\vec{a}$ va $\vec{b}$ vektorlar qarama-qarshi vektorlar deyiladi va $\vec{a} = -\vec{b}$ deb yoziladi.

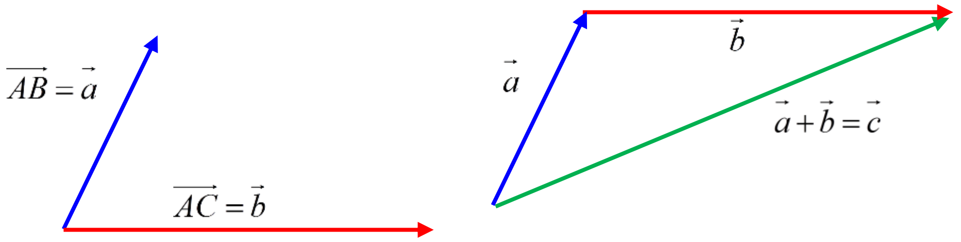


3-chizma

Vektorlar ustida chiziqli amallar.

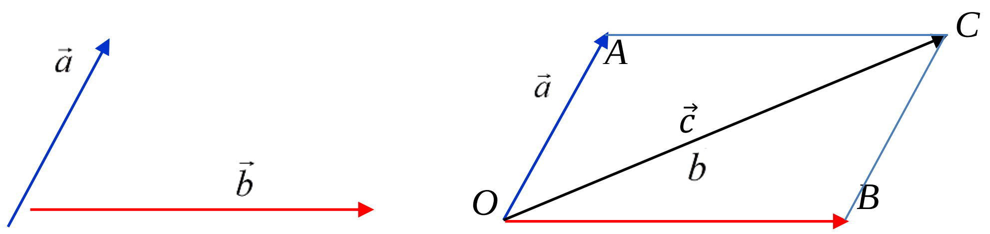
1) Vektorlarni qo'shish va ayrish. Vektorlar o'z-o'ziga parallel ko'chirilsa, berilgan vektorga teng vektor hosil bo'ladi. Ikkita $\vec{a}$ va $\vec{b}$ vektorning yig'indisini topish uchun

$\vec{a} = \overrightarrow{OA}$ vektorning oxiri $\vec{b}$ vektorning boshi bilan ustma-ust tushadigan qilib $\vec{b}$ vektorni o‘z-o‘ziga parallel ko‘chiramiz. Hosil bo‘lgan vektorni $\vec{b} = \overrightarrow{AB}$ deb belgilaymiz (4-chizma). O nuqta bilan B nuqtani tutashtiramiz. Natijada hosil bo‘lgan $\overrightarrow{OB} = \vec{c}$ vektor $\vec{a}$ va $\vec{b}$ vektorlarning yig‘indisi deyiladi va $\vec{c} = \vec{a} + \vec{b}$ kabi yoziladi. Vektorlarni bunday qo‘shish qoidasi «*uchburchak qoidasi*» deb ataladi(4-chizma).



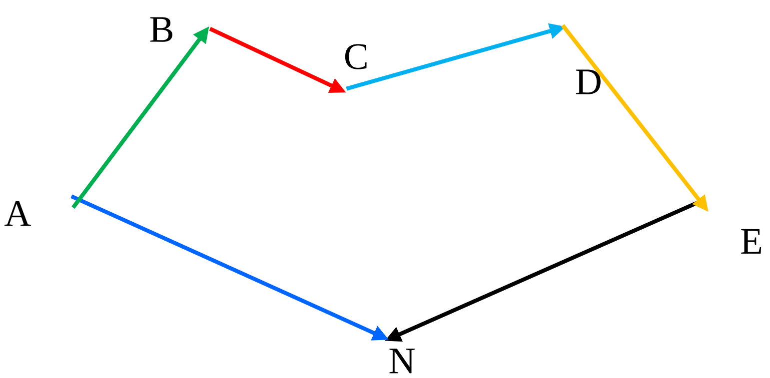
4-chizma

$\vec{a}, \vec{b}$ vektorlar o‘zaro kollinear bo‘lmagan vektor bo‘lsin. Ularning boshini bitta O nuqtaga o‘z-o‘ziga parallel ravishda ko‘chiramiz, so‘ngra tomonlari $\vec{a}$ va $\vec{b}$ vektorlardan iborat parallelogramm chizamiz. Uning O nuqtaga qarama-qarshi uchini C deb $\overrightarrow{OC}$ vektorni qaraymiz. Ravshanki, $\overrightarrow{OC} = \vec{c} = \vec{a} + \vec{b}$. Vektorlar yig‘indisini bunday geometrik yasashga odatda «*parallelogramm qoidasi*» deb yuritiladi (5-chizma).



5-chizma.

Bizga bir necha $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DE}, \overrightarrow{EN}$ vektorlar berilgan bo‘lsin. Bu vektorlarning har biri ketma-ket kelgan jufti uchun birinchisining oxiri bilan ikkinchisining boshi ustma-ust tushsin (6-chizma). Bu holda vektorlar siniq chiziq tashkil qilib, yig‘indi vektor ularning yopuvchisiga teng, ya’ni $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EN} = \overrightarrow{AN}$



6-chizma

$\vec{a},\vec{b}$ vektorlarning ayirmasi deb shunday $\vec{x}$ vektorga aytiladki, uni $\vec{b}$ vektorga qo‘shganda $\vec{a}$ vektor hosil bo‘ladi, ya’ni agar $\vec{x}$ vektor uchun ushbu $\vec{x}+\vec{b}=\vec{a}$ munosabat o‘rinli bo‘lsa, u holda $\vec{x}$ vektor $\vec{a}$ va $\vec{b}$ vektorlarning ayirmasi deyiladi hamda $\vec{x}=\vec{a}-\vec{b}$ deb yoziladi.

Agar «kamayuvchi» $\vec{a}$ va «ayriluvchi» $\vec{b}$ vektorlar berilsa, u holda ushbu $\vec{b}+\vec{x}=\vec{a}$ munosabatni qanoatlantiruvchi $\vec{x}$ vektor doim mavjud. $\overrightarrow{BC}=\vec{x}$, $\overrightarrow{AC}=\vec{a}$, $\overrightarrow{AB}=\vec{b}$. Demak, $\vec{a}-\vec{b}$ ayirma vektorni chizish uchun bir nuqtadan chiquvchi $\vec{a}$ va $\vec{b}$ vektorlarni chizib, $\vec{b}$ vektorning uchidan $\vec{a}$ vektorning uchiga boruvchi vektorni chizish kifoya. Shunday qilib, vektorlarni ayirish amali hamma vaqt ma’noga ega.

2) Vektorni songa ko‘paytirish.

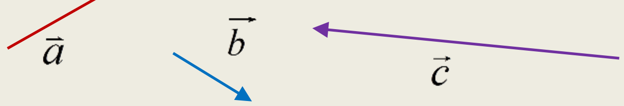
$\vec{a}$ vektorni $\lambda \in R$ soniga ko‘paytmasi deb shunday $\vec{b}$ vektorga aytiladiki, bu vektorning uzunligi $|\vec{b}|=\lambda\cdot|\vec{a}|$ teng bo‘lib, yo‘nalishi esa $\lambda>0$ bo‘lganda $\vec{a}$ vektor bilan bir xil yo‘nalgan, $\lambda<0$ bo‘lganda $\vec{a}$ vektorga qarama-qarshi yo‘nalgan bo‘ladi.

1-misol.

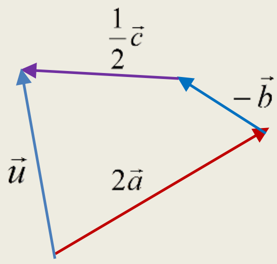
Berilgan $\vec{a}$, $\vec{b}$ va $\vec{c}$ vektorlarga asosan quyidagi vektorni yasang:

$$\vec{u}=2\vec{a}-\vec{b}+\frac{1}{2}\vec{c}.$$

►



Bir nuqtadan boshlab $2\vec{a}$, $-\vec{b}$ va $\frac{1}{2}\vec{c}$ vektorlarni ketma- ket joylashtiramiz:



Izlangan $2\vec{a}$ vektor hosil bo‘ladi,◄

Vektorning koordinatalari. Musbat yo‘nalishi tanlab olingan l to‘g‘ri chiziq o‘q deb ataladi. O‘qning yo‘nalishini odatda strelka bilan ko‘rsatiladi (7-chizma), bu strelkaning yo‘nalishi l to‘g‘ri chiziqdagi munosabat yo‘nalishni aniqlovchi $\vec{e}$ vektor yo‘nalishi bilan bir xil bo‘ladi.

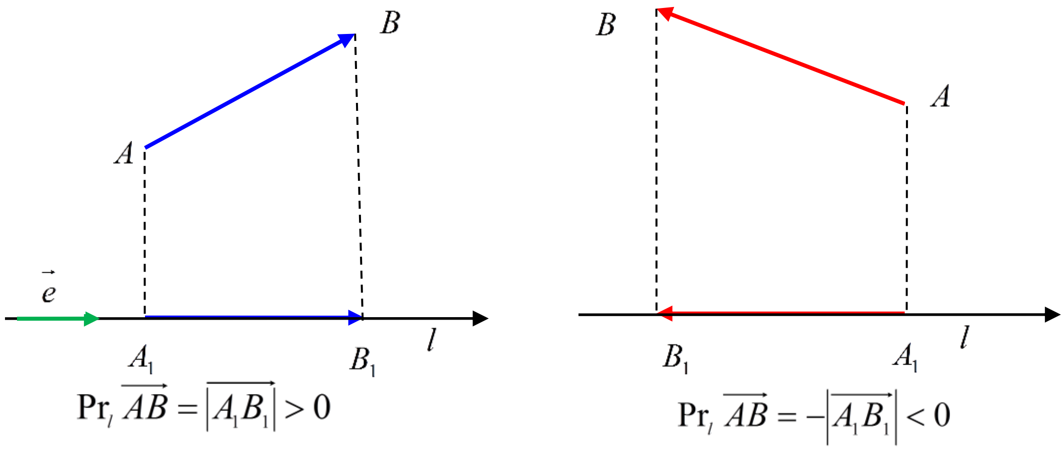
$\vec{e}$

7-chizma.

$$\overrightarrow{OE}=\vec{e}, \qquad \left|\overrightarrow{OE}\right|=|\vec{e}|=1.$$

Yo‘nalish o‘qdagi musbat yo‘nalish bilan bir xil bo‘lgan hamda uzunligi birga teng bo‘lgan vektor ($\vec{e}$ vektor) o‘qning *orti (bazisi)* deyiladi.

$\overrightarrow{AB}$ vektorning l o‘qdagi proeksiyasi deb, shunday $\overrightarrow{A_1B_1}$ vektorning uzunligiga aytiladiki, unda A_1 va B_1 lar mos ravishda A va B nuqtalarning l o‘qdagi ortogonal proeksiyalari bo‘lib, bu uzunlik $\overrightarrow{A_1B_1}$ va $\vec{e}$ vektorlarning yo‘nalishlari bir xil bo‘lganda musbat ishora bilan, aks holda manfiy ishora bilan olinadi (8-chizma).



8-chizma.

$\overrightarrow{AB}$ vektorning l o‘qdagi proeksiyasini $Pr_l\overrightarrow{AB}=\pm\left|A_1B_1\right|$. (5.1)

Bundan $\overrightarrow{AB}$ vektor o‘qqa perpendikulyar bo‘lgandagina uning proeksiyasi nolga teng degan xulosa kelib chiqadi. $\overrightarrow{A_1B_1}=x\cdot\vec{e}$ tenglikdagi x son $\overrightarrow{AB}$ vektorning proeksiyasidir, ya’ni $x=Pr_l\overrightarrow{AB}$.

Vektorning o‘qdagi proeksiyasining xossalari:

1. $Pr_l\left(\vec{a}+\vec{b}+\vec{c}+\ldots+\vec{d}\right)=Pr_l\vec{a}+Pr_l\vec{b}+Pr_l\vec{c}+\ldots+Pr_l\vec{d}$
2. $Pr_l\left(\lambda\cdot\vec{a}\right)=\lambda\cdot Pr_l\vec{a}$, $\lambda\neq 0$.
3. Teng vektorlarning bitta o‘qqa proeksiyalari o‘zaro tengdir.
4. $Pr_l\vec{a}=|\vec{a}|\cdot cos\varphi$, bu yerda φ - $\vec{a}$ va $\vec{e}$ vektorlar orasidagi burchak, $0\leq\varphi\leq\pi$.

Agar tekislikda (yoki fazoda) koordinatalar boshi deb ataluvchi nuqta, o‘zaro perpendikulyar to‘g‘ri chiziqlar, ularda musbat yo‘nalish hamda uzunlik birligi (umuman aytganda, har bir yo‘nalishdagi o‘qda har xil) tanlangan bo‘lsa, tekislikda (fazoda) *Dekart koordinatalar sistemasi* berilgan deyiladi. O‘qlar mos ravishda abssissalar o‘qi, ordinatalar o‘qi, (aplikatalar o‘qi) deb yuritiladi. Tegishli o‘qlar koordinatalar o‘qlari deyiladi. Faraz qilaylik, tekislikda Dekart koordinatalar sistemasi berilgan bo‘lsin (uni qisqacha *Oxy* sistema deb ham yuritiladi) va $\vec{a}$ vektor koordinatalar boshi *O* nuqtadan chiqqan bo‘lsin.

$\vec{a}$ *vektorning koordinatalari* deb uning koordinata o‘qlaridagi proeksiyalariga aytiladi, ya’ni

$$x = Pr_{Ox}\vec{a}\ , \quad y = Pr_{Oy}\vec{a}\,.$$

Agar Oxy sistemada $\vec{a} = \{x_1, y_1\}$, $\vec{b} = \{x_2, y_2\}$ bo‘lsa, $\vec{a} + \vec{b} = \vec{c}\{x_1 + x_2,\ y_1 + y_2\}$ bo‘ladi.

Agar *Oxy* sistemada $\vec{a}$ vektorning koordinatalari $\{x, y\}$ bo‘lsa, $\lambda \cdot \vec{a}$ vektorning shu sistemadagi koordinatalari $\{\lambda x, \lambda y\}$ bo‘ladi.

Agar *Oxy* sistemada $\overrightarrow{AB}$ vektor boshining koordinatalari $\{x_1, y_1\}$ va oxiri $\{x_2, y_2\}$ bo‘lsa, $\overrightarrow{AB}$ vektorning koordinatalari $\{x_2 - x_1\ ,\ y_2 - y_1\}$ bo‘ladi , ya’ni

$$\overrightarrow{AB} = \{x_2 - x_1\ , y_2 - y_1\} \tag{5.2}$$

2-misol

Agar $\vec{a}\{5,4\}$ vektor boshining koordinatalari $A(-2,3)$ bo‘lsa, uning oxirining koordinatalarini aniqlang.

► $\vec{a}\{5,4\}$ vektor oxirining koordinatalari $B(x, y)$ bo‘lsin. U holda $x - (-2) = 5,\ y - 3 = 4 \Leftrightarrow x = 5 - 2 = 3,\ \ y = 4 + 3 = 7$ bo‘ladi. Demak, $B(3,7)$.◄

3-misol

Agar $\vec{b}\{2, -1\}$ vektor oxirining koordinatalari B(3,2) bo‘lsa, uning boshining koordinatalarini aniqlang.

► $\vec{b}\{2, -1\}$ dan

$$3 - x = 2, \quad 2 - y = -1, \quad x = 3 - 2 = 1, \quad y = 2 + 1 = 3.$$

Bundan $A(1,3)$.◄

2.1.1 Chiziqli bog‘liq va chiziqli erkli vektorlar sistemasi. Bazis.

Bizga n ta $\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{a_3}, \ldots, \overrightarrow{a_n}$ vektorlar va n ta $\alpha_1,\ \alpha_2,\ \alpha_3, \ldots, \alpha_n$ sonlar berilgan bo‘lsin bu sonlarning mos vektorlarga ko‘paytmalarining yig‘indisini tuzamiz. $\alpha_1 \cdot \overrightarrow{a_1} + \alpha_2 \cdot \overrightarrow{a_2} + \alpha_3 \cdot \overrightarrow{a_3} + \ldots + \alpha_n \cdot \overrightarrow{a_n}$ ko‘paytmaga vektorlar sistemasining chiziqli kombinatsiyasi deyiladi.

$\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{a_3},\ldots,\overrightarrow{a_n}$ vektorlar sistemasi uchun kamida bittasi noldan farqli shunday n ta α_1 , α_2 , α_3 ,…, α_n sonlar mavjud bo'lsaki, ular uchun vektorlar sistemasining chiziqli kombinatsiyasi nolga teng, ya'ni

$$\alpha_1\cdot\overrightarrow{a_1}+\alpha_2\cdot\overrightarrow{a_2}+\alpha_3\cdot\overrightarrow{a_3}+\ldots+\alpha_n\cdot\overrightarrow{a_n}=0 \qquad (5.3)$$

bo'lsa, bunday vektorlar sistemasiga *chiziqli bog'liq sistema* deb ataladi. Aks holda $\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{a_3},\ldots,\overrightarrow{a_n}$ vektorlar *chiziqli erkli* deyiladi, ular uchun (5.3) tenglik faqat $\alpha_1=\alpha_2=\alpha_3=\ldots=\alpha_n=0$ bo'lgandagina o'rinli bo'ladi.

Agar vektorlar chiziqli bog'liq bo'lsa, (5.3) dagi biror vektorni boshqa vektorlar orqali ifodalab olish mumkin. $\alpha_1\cdot\overrightarrow{a_1}$ ifodani qoldirib qolgan ifodalarni tenglikning o'ng tomoniga o'tkazib $\alpha_1\neq 0$ ga bo'lsak,

$$\vec{a}_1=-\frac{\alpha_2}{\alpha_1}\cdot\overrightarrow{a_2}-\frac{\alpha_3}{\alpha_1}\cdot\overrightarrow{a_3}-\frac{\alpha_4}{\alpha_1}\cdot\overrightarrow{a_4}-\ldots--\frac{\alpha_n}{\alpha_1}\cdot\overrightarrow{a_n}$$

va belgilash kiritsak, bu vektor qolgan vektorlarning chiziqli kombinatsiyasidan iborat bo'ladi:

$$\overrightarrow{a_1}=\beta_2\cdot\overrightarrow{a_2}+\beta_3\cdot\overrightarrow{a_3}+\beta_4\cdot\overrightarrow{a_4}+\ldots+\beta_n\cdot\overrightarrow{a_n}\ . \qquad (5.6)$$

Agar vektorlardan kamida biri qolgan vektorlarning chiziqli kombinatsiyasidan iborat bo'lsa, u holda bu vektorlar chiziqli bog'liqdir. Aks holda barcha vektorlar chiziqli erkli bo'ladi.

Ixtiyoriy $\vec{a}$ vektorni n ta chiziqli erkli $\overrightarrow{e_1},\ \overrightarrow{e_2},\ \overrightarrow{e_3},\ldots,\overrightarrow{e_n}$ vektorlarning chiziqli kombinasiyasi ko'rinishida ifodalash mumkin bo'lsa, u holda shu n ta vektorlar fazoning *bazisi* deyiladi.

Bazisni hosil qiladigan vektorlar soni *fazoning o'lchami* deb ataladi. Bazisga kiruvchi vektorlar *bazis vektorlar* deb ataladi.

1. To'g'ri chiziqning o'lchami 1 ga teng, chunki to'g'ri chiziqda istalgan $\vec{e}$ vektor bazis hosil qiladi, qolgan vektorlar shu bazis vektor orqali ifodalanadi:

$$\vec{a}=\alpha\cdot\vec{e}\ ,\qquad \alpha\neq 0\ . \quad \text{(1 o'lchovli fazo)}$$

2. Tekislikning o'lchami 2 teng, chunki tekislikda kollinear bo'lmagan istalgan ikkita $\overrightarrow{e_1}$ va $\overrightarrow{e_2}$ vektor chiziqli erkli bo'lib, bazis hosil qiladi, qolgan vektorlarni esa ular orqali ushbu ko'rinishda ifodalash mumkin:

$$\vec{a}=\alpha\cdot\overrightarrow{e_1}+\beta\cdot\overrightarrow{e_2}\ ,\quad \left(\alpha^2+\beta^2\neq 0\right). \qquad \text{(2 o'lchovli fazo)}$$

3. Fazoda

$$\vec{a}=\alpha\cdot\vec{e}_1+\beta\cdot\vec{e}_2+\gamma\cdot\vec{e}_3\ ,\ \left(\alpha^2+\beta^2+\gamma^2\neq 0\right). \quad \text{(3 o'lchovli fazo)}$$

Vektorlarni bazis vektorlarning chiziqli kombinatsiyasi ko'rinishida ifodalashga *bazis bo'yicha yoyish* deyiladi.

Ba'zis vektorning uzunliklari har xil bo'ladi Biz amaliyotda birlik uzunlikka ega bo'lgan birlik vektorlardan tashkil topgan bazislar bilan shug'ullanamiz. Bazis vektorlar bir biriga nisbatan har xil joylashgan (har xil burchak ostida) bo'ladi. Biz koordinata o'qlarida yotuvchi, yo'nalishi koordinata o'qlarining musbat yo'nalishi bilan ustma-ust tushuvchi birlik uzunlikka ega bo‘lgan va o‘zaro perpendikulyar bo‘lgan $\vec{i}$, $\vec{j}$, $\vec{k}$ birlik bazis vektorlar bilan shug‘ullanamiz Bu vektorlar *ortonormal vektorlar* yoki *ortlar* deyiladi.

$\vec{a}=\overrightarrow{OA}$ vektorning o‘qlaridagi proeksiyalari mos ravishda a_x, a_y, a_z bilan belgilasak, uning birlik–bazis vektorlar(ortlar) orqali yozuvi

$$\vec{a}\left(a_x, a_y, a_z\right)=a_x\vec{i}+a_y\vec{j}+a_z\vec{k} \qquad (5.7)$$

dan iborat bo‘ladi.

Bu ifodaga $\vec{a}$ vektorning $\vec{i}$, $\vec{j}$, $\vec{k}$ ba’zis vektorlar yoki koordinata o‘qlari bo‘yicha yoyilmasi deyiladi

Koordinata boshidan chiqqan vektorga *radius vektor* deyiladi.

4-misol

Agar $\vec{a}\{-1,4\}$, $\vec{b}\{2,-1\}$, $\vec{c}\{3, 5\}$ vektorlar koordinatalari bilan berilgan bo‘lsa quyidagi vektorlarning koordinatalari aniqlansin:

$$a)\ \frac{\vec{c}-2\vec{b}}{2},\quad b)\ \frac{\vec{a}+\vec{b}}{2}-\vec{c}.$$

► $$a)\frac{\vec{c}-2\vec{b}}{2}=\vec{d}\left\{\frac{3-2\cdot 2}{2},\ \frac{5-2\cdot(-1)}{2}\right\}=\vec{d}\{-0.5, 3.5\},$$

$$b)\frac{\vec{a}+\vec{b}}{2}-\vec{c}=\vec{s}\left\{\frac{-1+2}{2}-3,\ \frac{4+(-1)}{2}-5\right\}=\vec{s}\{-2.5, -3.5\}.$$ ◄

Auditoriya topshiriqlari

1. Agar $\vec{c}=\overrightarrow{AB}$, $\vec{b}=\overrightarrow{AC}$ va $\vec{a}=\overrightarrow{BC}$ vektorlar ABC uchburchakning tomonlari bo‘lsa, u holda bu uchburchakning $\overrightarrow{AN}$, $\overrightarrow{BM}$ va $\overrightarrow{CP}$ medianalarini $\vec{a}$, $\vec{b}$ va $\vec{c}$ vektorlar orqali ifodalang.

2. $\vec{a}=\vec{i}+4\vec{j}-5\vec{k}$ va $\vec{b}=3\vec{i}-2\vec{j}+3\vec{k}$ vektorlar berilgan bo‘lsa, $\vec{u}=2\vec{a}-3\vec{b}$ va $\vec{v}=-\frac{3}{4}\vec{a}+\frac{1}{2}\vec{b}$ vektorlarni aniqlang. Dekart koordinatalar sistemasida $\vec{u}$ va $\vec{v}$ vektorlarni yasang.

3. $\vec{a}(2;-3;4)$, $\vec{b}(5;3;-2)$ vektorlarga qurilgan parellelogramning diagonallarini ifodalovchi vektorlarni toping.

4. $ABCD$ to‘g‘rito‘rtburchakning tomonlari uzunliklari $AB=4$, $BC=3$ bo‘lib, A va B uchidan $\overrightarrow{AB}$ va $\overrightarrow{BC}$ vektorlar yo‘nalishida $\vec{a}$ va $\vec{b}$ birlik vektorlar qo‘yilgan.

1) $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{AC}$, $\overrightarrow{CB}$ va $\overrightarrow{DB}$ vektorlarni $\vec{a}$ va $\vec{b}$ vektorlar orqali ifodalang.

2) N va P nuqtalar mos ravishda BC va CD tomonlarning o‘rtasi bo‘lsa, $\overrightarrow{AN}$, $\overrightarrow{AP}$ va $\overrightarrow{PN}$ vektorlarni $\vec{a}$ va $\vec{b}$ vektorlar orqali ifodalang.

5. Radiusi $R=3$ bo‘lgan aylananing $90°$ li AB yoyini C nuqta orqali $AC:CB=3:2$ nisbatda AC va CB yoylarga bo‘lingan. Agar $\overrightarrow{OA}=\vec{a}$ va $\overrightarrow{OB}=\vec{b}$ bo‘lsa, $\overrightarrow{OC}$ vektorni $\vec{a}$ va $\vec{b}$ vektorlar orqali ifodalang.

6. To‘g‘riburchakli $ABCD$ trapetsiyaning asoslari $AD=4,\ BC=2$ bo‘lib, D buchagi $45°$ ga teng. $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{BC}$ va $\overrightarrow{AD}$ vektorlarni $\overrightarrow{CD}$ vektor bilan aniqlangan l o‘qga proyeksiyalarini toping.

7. Asosi ychburchakdan iborat bo‘lgan SABC piramidada $\overrightarrow{SA}=\vec{a}$, $\overrightarrow{SB}=\vec{b}$ va $\overrightarrow{SC}=\vec{c}$. Agar M nuqta ΔABC ning og‘irlik markazi bo‘lsa, $\overrightarrow{SM}$ vektorni bu vektorlar orqali ifoda qiling.

8. Uchburchakning $A(1;2;-1)$ uchi, $\overrightarrow{AB}=\{-2;1;4\}$ va $\overrightarrow{BC}=\{3;-1;4\}$ tomonlari yotgan vektorlar berilgan bo‘lsa, uchburchakning qolgan uchlari va $\overrightarrow{AC}$ vektorni toping.

Mustaqil yechish uchun testlar

1. Agar $A(2;0;4)$, $B(5;2;4)$, $C(-2;6;5)$, $D(-5;6;3)$ berilgan bo‘lsa, $\vec{a}=\overrightarrow{AB}+\overrightarrow{CD}$ vektorni toping

$A)$ $\vec{a}(0;2;-2)$; B) $\vec{a}(5;14;10)$; $C)$ $\vec{a}(4;7;-2)$; $D)$ $\vec{a}(7;1;0)$

2. Agar $A(2;0;4)$, $B(5;2;4)$, $C(-2;6;5)$, $D(0;6;3)$ berilgan bo‘lsa, $\vec{a}=\overrightarrow{AB}-\overrightarrow{CD}$ vektorni toping

$A)$ $\vec{a}(6;2;2)$; B) $\vec{a}(0;-2;-2)$; $C)$ $\vec{a}(4;7;-2)$; $D)$ $\vec{a}(7;1;0)$

3. $A(1,-2,3)$, $B(3,4,-6)$ berilgan bo‘lsa, $\overrightarrow{AB}$ vektor uzunligini toping

A) 7; B) 11; C) 13; D) 8

4. $A(-4;0;2)$, $B(-1;2;-2)$, $C(6;-2;4)$ uchburchak uchlari koordinatalari bo‘lsa, mediana chizig‘ini ifodalovchi $\overline{BE}$ vektor koordinatalarini aniqlang

$A)$ $\{2;-3;5\}$; B) $\{2;3;-5\}$; $C)$ $\{-2;3;-5\}$; $D)$ $\overline{a}(7;1;0)$

5. $\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{a_3},\dots,\overrightarrow{a_n}$ vektorlarning … bittasini qolganlarining chiziqli kombinatsiyasi shaklida ifodalash mumkin …, bu sistema chiziqli bog‘liq sistema bo‘ladi.

A) kamida, bo‘lsa; B) ixtiyoriy, bo‘lsa;
D) kamida, bo‘lmasa; D) ixtiyoriy, bo‘lmasa;

6. $\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{a_3},\dots,\overrightarrow{a_n}$ vektorlarning … bittasini qolganlarining chiziqli kombinatsiyasi shaklida ifodalash mumkin …, bu sistema chiziqli erkli sistema bo‘ladi

A) kamida, bo‘lsa; B) ixtiyoriy, bo‘lsa;
D) kamida, bo‘lmasa; D) ixtiyoriy, bo‘lmasa;

2.2 Kesmani berilgan nisbatda bo‘lish. Vektorlarning skalyar ko‘paytmasi

2.2.1 Ikki nuqta orasidagi masofa. Kesmani berilgan nisbatda bo‘lish

a) *Ikki nuqta orasidagi masofa.* Fazoda ikkita ixtiyoriy $A(x_1, y_1, z_1),\ B(x_2, y_2, z_2)$ nuqta berilgan bo‘lsin. Bu nuqtalar orasidagi masofani topish bilan shug‘ullanamiz. A, B nuqtalarni koordinatalar boshi O nuqta bilan tutashtirib, bu nuqtalarning radius-vektorlarini yasaymiz. Izlanayotgan masofani $d(A,B)$ bilan belgilaymiz, ya’ni $\left|\overrightarrow{AB}\right| = d(A,B)$. Bu holda $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ $\overrightarrow{OA}$ va $\overrightarrow{OB}$ radius-vektorlarning koordinatalari mos ravishda $\overrightarrow{OA} = \{x_1, y_1, z_1\}$, $\overrightarrow{OB} = \{x_2, y_2, z_2\}$ bo‘lgani uchun $\overrightarrow{AB}$ vektorning to‘g‘ri burchakli $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ bazisga nisbatan koordinatalari quyidagicha bo‘ladi:

$$\overrightarrow{AB} = \{x_2 - x_1\ , y_2 - y_1\ , z_2 - z_1\} \iff \overrightarrow{AB} = (x_2 - x_1)\cdot\vec{e}_1 + (y_2 - y_1)\cdot\vec{e}_2 + (z_2 - z_1)\cdot\vec{e}_3$$

bundan

$$\left|\overrightarrow{AB}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{6.1}$$

ni hosil qilamiz. $\left|\overrightarrow{AB}\right|$ esa A va B nuqtalar orasidagi $d(A,B)$ masofa bo‘lgani uchun

$$d(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Agar tekislikda ikkita $A(x_1, y_1),\ B(x_2, y_2)$ nuqta berilgan bo‘lsa, ular orasidagi masofa

$$d(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{6.2}$$

formula bilan aniqlanadi.

b) *Kesmani berilgan nisbatda bo‘lish.* $A(x_1, y_1, z_1)$ va $B(x_2, y_2, z_2)$ nuqtalar fazodagi ikkita ixtiyoriy har xil nuqta bo‘lsin.

A va B nuqtalardan o‘tuvchi to‘g‘ri chiziqning ixtiyoriy nuqtasi C uchun

$$\overrightarrow{AC} = \lambda \cdot \overrightarrow{CB} \tag{6.3}$$

tenglik o‘rinli. (5.5) da C nuqta $[AB]$ kesmaning ichki nuqtasi bo‘lsa, $\lambda > 0$, C nuqta $[AB]$ kesmaning tashqi nuqtasi bo‘lsa, $\lambda < 0$ bo‘ladi.

$[AB]$ kesmani berilgan nisbatda bo‘lish masalasi quyidagicha aniqlanadi: $A(x_1, y_1, z_1)$ va $B(x_2, y_2, z_2)$ nuqtalar va λ son berilgan. (A, B) to‘g‘ri chiziqda yotuvchi va (6.3) tenglikni qanoatlantiruvchi C nuqtaning koordinatalari topilsin.

Ravshanki, (6.3) dan $\left|\overrightarrow{AC}\right| = |\lambda| \cdot \left|\overrightarrow{CB}\right|$, bundan

$$|\lambda| = \frac{\left|\overrightarrow{AC}\right|}{\left|\overrightarrow{CB}\right|} \tag{6.4}$$

Shuning uchun biz qarayotgan masala (AB) to‘g‘ri chiziqda yotib, $[AB]$ kesmani $\lambda > 0$ bo‘lganda ichkarida, $\lambda < 0$ bo‘lganda tashqaridan $\lambda : 1$ nisbatda bo‘luvchi C nuqtaning koordinatalarini topishdan iboratdir.

C nuqtaning Dekart koordinatalarini $\{x, y, z\}$ bilan belgilaylik. U holda (6.3) tenglikka ko‘ra ushbu tengliklar sistemasini hosil qilamiz:

$$x - x_1 = \lambda \cdot (x_2 - x),\ \ y - y_1 = \lambda \cdot (y_2 - y),\ \ z - z_1 = \lambda \cdot (z_2 - z)$$

$\lambda \neq -1$ ekanini hisobga olib, C nuqtaning koordinatalari uchun bundan quyidagi formulalarni hosil qilamiz:

$$x = \frac{x_1 + \lambda \cdot x_2}{1+\lambda},\ y = \frac{y_1 + \lambda \cdot y_2}{1+\lambda},\quad z = \frac{z_1 + \lambda \cdot z_2}{1+\lambda} \tag{6.5}$$

Agar $\lambda = 1$ bo‘lsa, (5.6) dan ushbu formulaga ega bo‘lamiz.

$$x = \frac{x_1 + x_2}{2},\quad y = \frac{y_1 + y_2}{2},\quad z = \frac{z_1 + z_2}{2} \tag{6.6}$$

Bu berilgan kesma o‘rtasining koordinatalarini beradi. Agar $[AB]$ kesma tekislikda berilgan bo‘lsa, uni λ nisbatda bo‘lish formulalari

$$x = \frac{x_1 + \lambda \cdot x_2}{1+\lambda},\quad y = \frac{y_1 + \lambda \cdot y_2}{1+\lambda}$$

ko‘rinishda bo‘ladi.

1-misol

Oxirgi nuqtalari $A(-1; 8; -3)$ va $B(9; -7; 2)$ bo‘lgan kesma P_1, P_2, P_3 va P_4 nuqtalar bilan teng beshta bo‘lakka bo‘lingan bo‘lsa P_1 va P_3 nuqtalarning koordinatalarini toping.

► $AP_1 : P_1B = 1:4$ bo‘lgani uchun, $\lambda = \dfrac{1}{4}$. (6.5) formulaga ko‘ra, $P_1(x_1;\ y_1; z_1)$ koordinatalari

$$x_1 = \frac{4\cdot(-1)+9}{4+1} = 1,\ \ y_1 = \frac{4\cdot 8 + (-7)}{4+1} = 5,\ \ z_1 = \frac{4\cdot(-3)+2}{4+1} = -2.$$

$AP_3 : P_3B = 3:2$ bo‘lgani uchun, $\lambda = \dfrac{3}{2}$. (6.5) formulaga ko‘ra, $P_3(x_3;\ y_3; z_3)$ koordinatalari

$$x_3 = \frac{2\cdot(-1)+3\cdot 9}{2+3} = 5,\ \ y_1 = \frac{2\cdot 8 + 3\cdot(-7)}{2+3} = -1,\ \ z_1 = \frac{2\cdot(-3)+3\cdot 2}{2+3} = 0.$$

Demak, $P_1(1; 5; -2)$ va $P_3(5; -1; 0)$.◄

2.2.2 Vektorlarni skalyar ko‘paytirish

Ikki $\vec{a}$ va $\vec{b}$ *vektorning skalyar ko‘paytmasi* deb, bu vektorlar uzunliklarini ular orasidagi burchak kosinusi bilan ko‘paytmasiga teng bo‘lgan songa aytiladi va $(\vec{a},\vec{b})$ yoki $\vec{a}\cdot\vec{b}$ bilan belgilanadi.

Ta’rifga ko‘ra,

$$(\vec{a},\vec{b})=\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot cos\varphi \tag{6.7}$$

Skalyar ko‘paytma tushunchasining manbai mexanikadir. Haqiqatan, agar $\vec{a}$ ozod vektor qo‘yilgan nuqta $\vec{b}$ vektorning boshidan oxiriga siljuvchi kuchni tasvirlasa, bu kuch bajargan A ish ushbu tenglik bilan aniqlanadi:

$$A=|\vec{a}|\cdot|\vec{b}|\cdot cos\varphi$$

2-misol

Berilgan $\vec{F}=\{6,-2,1\}$ kuchning to‘g‘ri chiziq bo‘ylab $A(3,4,-2)$ nuqtadan $B(4,-2,-3)$ nuqtaga siljishida bajargan ishni hisoblang

► $\overrightarrow{AB}=\{x,y,z\}$ vektorning koordinatalarini aniqlaymiz. Buning uchun $x=x_B-x_A,\ y=y_B-y_A,\ z=z_B-z_A$ formulalarga A va B nuqtalarning koordinatalarini qo‘yib $x=4-3=1,\ y=-2-4=-6,\ z=-3+2=-1$ larni topamiz.

Demak, $\overrightarrow{AB}=\{1,-6,-1\}$. $\vec{F}$ kuch ta’siri ostida bajarilgan ish o‘tilgan $\overrightarrow{AB}$ yo‘l bilan $\vec{F}$ kuchning skalyar ko‘paytmasiga tengligidan, ya’ni ish $\vec{F}\cdot\overrightarrow{AB}$ ga teng. Shuni hisoblaymiz:

$\vec{F}\cdot\overrightarrow{AB}=\left(6\vec{e}_1-2\vec{e}_2+\vec{e}_3\right)\cdot\left(\vec{e}_1-6\vec{e}_2-\vec{e}_3\right)=6\cdot1+(-2)\cdot(-6)+\cdot(-1)=6+12-1=17.$

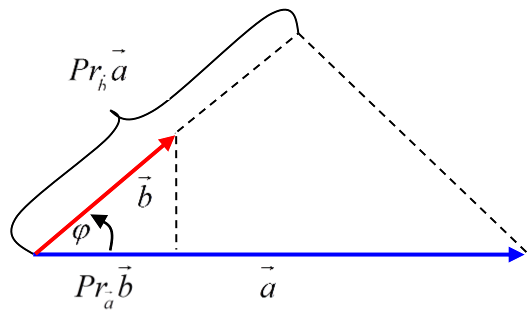
Demak, $A=\vec{F}\cdot\overrightarrow{AB}=17$. ◄

Agar $\vec{a}\cdot\vec{b}$ ko‘paytmani $|\vec{a}|\cdot|\vec{b}|\cdot cos\varphi$ ko‘rinishda yozib, $|\vec{b}|\cdot cos\varphi=Pr_{\vec{a}}\vec{b}$ ekanini e’tiborga olsak, $\vec{a}\cdot\vec{b}=|\vec{a}|\cdot Pr_{\vec{a}}\vec{b}$ ni hosil qilamiz.

$|\vec{a}|\cdot cos\varphi=Pr_{\vec{b}}\vec{a}$ ekanligini e’tiborga olsak, $\vec{a}\cdot\vec{b}=|\vec{b}|\cdot Pr_{\vec{b}}\vec{a}$ ni hosil qilamiz. Demak,

$$\vec{a}\cdot\vec{b}=|\vec{a}|\cdot Pr_{\vec{a}}\vec{b}=|\vec{b}|\cdot Pr_{\vec{b}}\vec{a} \tag{6.8}$$

formulalar o‘rinli. Boshqacha aytganda, ikki vektorning skalyar ko‘paytmasi ulardan birining uzunligi miqdori bilan ikkinchisining shu vektor yo‘nalishidagi proeksiysi ko‘paytmasiga teng.



1-chizma.

Agar ikki vektor orasidagi burchak $\frac{\pi}{2}$ ga teng bo‘lsa, ular *ortogonal vektorlar* deyiladi.

3-misol

Agar $\vec a, \vec b$ va $\vec c$ vektorlar koordinatalari bilan berilgan, ya’ni:

$$\vec a = \vec i - 4\vec j + 8\vec k\ ;\ \vec b = 4\vec i + 4\vec j - 2\vec k\ ;\ \vec c = 2\vec i + 3\vec j + 6\vec k\ .$$

bo‘lsa $\left(\vec b + \vec c\right)$ vektorning $\vec a$ vektordagi proyeksiyasini toping.

▶ $\vec b + \vec c = 6\vec i + 7\vec j + 4\vec k = \vec d$, (6.8) dan $\Pr_{\vec a}(\vec b + \vec c) = \Pr_{\vec a}\vec d = \dfrac{\vec d \cdot \vec a}{|\vec a|}$ formulani hosil qilamiz. $\Pr_{\vec a}(\vec b + \vec c) = \dfrac{6\cdot 1 + 7\cdot(-4) + 4\cdot 8}{\sqrt{1^2 + (-4)^2 + 8^2}} = \dfrac{10}{9}$. ◀

Skalyar ko‘paytmaning bir qator eng sodda xossalarini keltiramiz.

1-teorema. *Agar $\vec a \cdot \vec b = 0$ bo‘lsa, u holda $\vec a$ va $\vec b$ vektorlar ortogonal bo‘ladi.*

2-teorema. *Har qanday vektorning shu vektorning o‘ziga skalyar ko‘paytmasi bu vektorning uzunligi kvadratiga teng, ya’ni*

$$\vec a \cdot \vec a = \left|\vec a\right|^2 \tag{6.9}$$

3-teorema. *Skalyar ko‘paytma o‘rin almashtirish qonuniga bo‘ysunadi, ya’ni ixtiyoriy ikki $\vec a$ va $\vec b$ vektorlar uchun $\vec a \cdot \vec b = \vec b \cdot \vec a$ munosabat o‘rinli.*

4-teorema. *Skalyar ko‘paytma skalyar ko‘paytuvchiga nisbatan gruppalash qonuniga bo‘ysunadi, ya’ni $\left(\lambda \cdot \vec a\right) \cdot \vec b = \vec a \cdot \left(\lambda \cdot \vec b\right) = \lambda \cdot \left(\vec a \cdot \vec b\right)$ munosabatlar o‘rinli.*

5-teorema. *Skalyar ko'paytma qo'shishga nisbatan taqsimot qonuniga bo'ysunadi, ya'ni ixtiyoriy uchta* $\vec{a}$, $\vec{b}$ *va* $\vec{c}$ *vektorlar uchun ushbu tenglik o'rinli:*

$$\left(\vec{a}+\vec{b}\right)\cdot\vec{c}=\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}$$

Skalyar ko'paytmaning Dekart koordinatalar sistemasidagi formulasi

6-teorema. *Dekart koordinatalar sistemasida* $\vec{a}=\{x_1,y_1,z_1\}$ *va* $\vec{b}=\{x_2,y_2,z_2\}$ *vektorlar berilgan bo'lsa, bu vektorlarning skalyar ko'paytmasi ularning mos koordinatalar ko'paytmalarining yig'indisiga teng, ya'ni*

$$\vec{a}\cdot\vec{b}=x_1\cdot x_2+y_1\cdot y_2+z_1\cdot z_2 \tag{6.10}$$

Agar $\vec{a}=\{x_1,y_1\}$ *va* $\vec{b}=\{x_2,y_2\}$ *bo'lsa,*

$$\vec{a}\cdot\vec{b}=x_1\cdot x_2+y_1\cdot y_2 \tag{6.11}$$

bo'ladi.

$\vec{a}=\{x_1,y_1\}$ vektorning uzunligi koordinatalarda

$$\left|\vec{a}\right|=\sqrt{x^2+y^2} \tag{6.12}$$

$\vec{a}=\{x_1,y_1,z_1\}$ vektorning uzunligi esa

$$\left|\vec{a}\right|=\sqrt{x^2+y^2+z^2} \tag{6.13}$$

formuladan topiladi.

Vektorlar orasidagi burchak koordinatalari orqali (Dekart sisitemasida), ya'ni skalyar ko'paytma ta'rifiga ko'ra osongina topiladi: $\vec{a}=\{x_1,y_1\}$ va $\vec{b}=\{x_2,y_2\}$ vektorlar uchun

$$\cos\left(\vec{a},\vec{b}\right)=\frac{\vec{a}\cdot\vec{b}}{\left|\vec{a}\right|\cdot\left|\vec{b}\right|}=\frac{x_1\cdot x_2+y_1\cdot y_2}{\sqrt{x_1^2+y_1^2}\cdot\sqrt{x_2^2+y_2^2}} \tag{6.14}$$

$\vec{a}=\{x_1,y_1,z_1\}$ va $\vec{b}=\{x_2,y_2,z_2\}$ vektorlar uchun

$$\cos\left(\vec{a},\vec{b}\right)=\frac{\vec{a}\cdot\vec{b}}{\left|\vec{a}\right|\cdot\left|\vec{b}\right|}=\frac{x_1\cdot x_2+y_1\cdot y_2+z_1\cdot z_2}{\sqrt{x_1^2+y_1^2+z_1^2}\cdot\sqrt{x_2^2+y_2^2+z_2^2}} \tag{6.15}$$

formulalar o'rinli.

4-misol

Ikki $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak $\varphi=\pi/4$ ga teng va $\left|\vec{a}\right|=\sqrt{2}$, $\left|\vec{b}\right|=3$ ekanligi ma'lum bo'lsa $\vec{c}=2\vec{a}+3\vec{b}$ vektorning uzunligini hisoblang.

► $\vec{c}$ vektorning uzunligini topish uchun vektorlarning skalyar ko‘paytmasidan foydalanamiz. $\vec{a}\cdot\vec{a}=\vec{a}^{\,2}$ deb belgilab va $\vec{a}^{\,2}=\left|\vec{a}\right|^2$ ni e’tiborga olib, berilgan vektorning har ikki tomonini kvadratga ko‘taramiz:

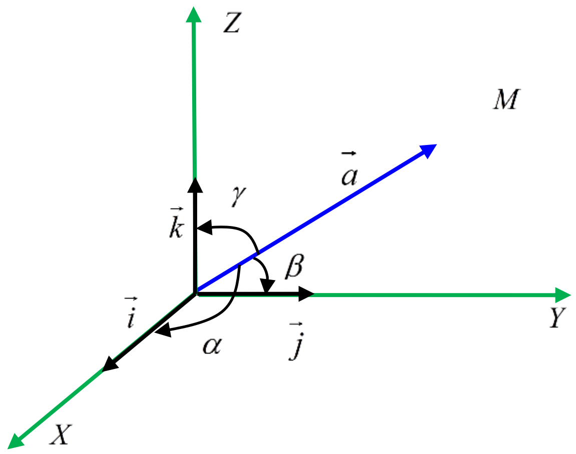
$$\vec{c}^{\,2}=\left(2\vec{a}+3\vec{b}\right)^2=4\vec{a}^{\,2}+12\vec{a}\cdot\vec{b}+9\vec{b}^{\,2}$$

berilganlarga asosan:

$$\vec{a}^{\,2}=\left|\vec{a}\right|^2=2;\ \ \vec{b}^{\,2}=\left|\vec{b}\right|^2=9;\ \ \vec{a}\cdot\vec{b}=\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot cos\varphi=\sqrt{2}\cdot 3\cdot\frac{\sqrt{2}}{2}=3.$$

Demak, $\vec{c}^{\,2}=4\cdot 2+12\cdot 3+9\cdot 9=125$ yoki $\left|\vec{c}\right|=\sqrt{125}=5\sqrt{5}$. ◄

Odatda vektorning koordinata o‘qlari bilan tashkil qilgan $\alpha,\ \beta,\ \gamma$ burchaklarning kosinuslari uning *yo‘naltiruvchi kosinuslari* deyiladi(2-chizma).



2-chizma.

$\vec{a}=\{x,y,z\}$ vektorning yo‘naltiruvchi kosinuslari uning koordinatalari orqali quyidagicha aniqlanadi:

$$\cos\alpha=\frac{x}{\sqrt{x^2+y^2+z^2}},\ \cos\beta=\frac{y}{\sqrt{x^2+y^2+z^2}},\ \cos\gamma=\frac{z}{\sqrt{x^2+y^2+z^2}} \qquad (6.16)$$

Birlik vektorlarning koordinatalari uning yo‘naltiruvchi kosinuslaridan iborat, ya’ni agar $\left|\overrightarrow{a^0}\right|=1$, bo‘lsa,

$$\overrightarrow{a^0}=\{\cos\alpha,\cos\beta,\cos\gamma\} \qquad (6.17)$$

(6.16) ga ko‘ra,

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1 \qquad (6.18)$$

formulani hosil qilish mumkin, ya’ni vektorning yo‘naltiruvchi kosinuslari kvadratlarining yig‘indisi birga teng.

Auditoriya topshiriqlari

1. $C(2;\, 0;\, 2)$ va $D(5;-2;\, 0)$ nuqtalar yordamida teng uch qismga bo‘lingan kesmaning oxirlari A va B nuqtalarning koordinatalarini toping.

Javob: $A(-1;\, 2;\, 4)$, $B(8;-4;\, 2)$

2. $\vec{a}$ va $\vec{b}$ vektorlar koordinatalari bilan berilgan:

$$\vec{a} = 7\vec{i} + 2\vec{j} + 3\vec{k}\ ;\ \ \vec{b} = 2\vec{i} - 2\vec{j} + 4\vec{k}$$

Bu vektorlarlarning skalyar ko‘paytmasini toping.

Javob: $\vec{a} \cdot \vec{b} = 22$

3. Agar $|\vec{a}| = 7\sqrt{2}$, $|\vec{b}| = 4$ va $(\vec{a},\wedge\vec{b}) = 45^0$ bo‘lsa, $3\vec{a} + \alpha\vec{b}$ va $\vec{a} - 2\vec{b}$ vektorlar α ning qanday qiymatlarida o‘zaro perpendikulyar bo‘ladi?

Javob: $\alpha = 31{,}5$

4. Uchlari $A(-1;5;1)$, $B(1;1;-2)$ va $C(-3;3;2)$ nuqtalarda bo‘lgan uchburchak berilgan. AC tomonni davom ettirishdan hosil bo‘lgan tashqi burchakni aniqlang.

Javob: $\varphi = arccos(4/9)$

5. Uchlari $\mathrm{A}(-2;3;1)$, $\mathrm{B}(-2;-1;4)$ va $C(-2;-4;0)$ nuqtalarda bo‘lgan uchburchak berilgan. Bu uchburchakning C ichki burchagini hisoblang.

Javob: $\angle BCA = \pi/4$

6. Agar $A(-4;0;4)$, $B(-1;2;-2)$, $C(6;-2;4)$ chburchak uchlari koordinatalari bo‘lsa, $\overline{BA}$ vektorni mediana chizig‘ini ifodalovchi $\overline{BE}$ vektorga proyeksiyasini aniqlang.

Javob: $5\dfrac{1}{7}$

7. Rombning tomonlari umumiy uchdan chiquvchi ***a*** va ***b*** vektorlarda joylashgan. Uning diagonallari perpendikulyar ekanligini isbotlang.

8. Agar $\overrightarrow{OA} = \vec{a}$ va $\overrightarrow{OB} = \vec{b}$ vektorlar berilgan hamda $|\vec{a}| = 2$, $|\vec{b}| = 4$ v va $(\vec{a},\wedge\vec{b}) = 60^0$ bo‘lsa, AOB uchburchakning $\overrightarrow{OA}$ tomoni $\overrightarrow{OM}$ medianasi orasidagi φ burchak kosinusini toping.

Javob: $\cos\phi = \dfrac{2}{\sqrt{7}}$

Mustaqil yechish uchun testlar

1. Agar $A(-4;1)$, $B(2;4)$ nuqtalar uchun $AC : CB = 2:1$ o‘rinli bo‘lsa, C-?

A) $C(-1;2)$; B) $\mathrm{C}(-1;3)$; C) $\mathrm{C}(0;3)$; D) $C(-2;2)$

2. Proyeksiyalar bilan berilgan ***a*** va ***b*** vektorlarning skalayar ko‘paytmasi qaysi javobda berilgan?

A) $|\vec{a}|Pr_{\vec{a}}\vec{b}$; B) $|\vec{b}|\Pr_{\vec{a}}\vec{b}$; D) $|\vec{a}|Pr_{\vec{b}}\vec{b}$ E) $Pr_{\vec{a}}\vec{b}\cdot Pr_{\vec{b}}\vec{a}$

3. $\vec{a}(2;1;6)$ va $\vec{b}(1;-2;-1)$ vektorlarning skalyar ko‘paytmasini hisoblang.

A) 0; B) -4; C) -6; D) 4

4. $\vec{a}(4;-7;4)$, $\vec{b}(4,-2,-3)$ vektorlar berilgan. U holda $pr_{\vec{a}}\vec{b}$ ni toping

A) 2; B) 3; C) 4; D) 5

5. Agar $|\vec{a}|=3, |\vec{b}|=2, (\vec{a},\wedge\vec{b})=60^0$ berilgan bo‘lsa, $(\vec{a}+\vec{b})\cdot(2\vec{a}-3\vec{b})$ skalyar ko‘paytma topilsin.

A) 2; B) 3; C) 6; D) 4

6. $A(1,-2,3)$, $B(3,4,-6)$, $C(-3,1,3)$ berilgan bo‘lsa, $\overrightarrow{AB}$ va $\overrightarrow{AC}$ vektorlar orasidagi burchak kosinusini toping

A) $\frac{1}{2}$; B) $\frac{2}{11}$; C) 1; D) 0

2.3 Vektorlarning vektor va aralash ko‘paytmalari

2.3.1 Ikki vektorning vektor ko‘paytmasi

Vektor ko‘paytma ta’rifini kiritishdan avval, biz uchta o‘zaro nokomplanar vektor uchligining fazoda joylashishi bilan bog‘liq bo‘lgan zarur bir tushunchani kiritamiz. Shuni aytib o‘tamizki, keyingi punktlarda yuritiladigan mulohazalar faqat uch o‘lchovli fazoga doir bo‘ladi.

Agar komplanar $\vec{a}$, $\vec{b}$ va $\vec{c}$ vektorlar boshi umumiy nuqtaga keltirilgandan so‘ng $\vec{n}$ vektorning oxiridan (uchidan) qaraganda $\vec{a}$ vektordan $\vec{b}$ vektoga qarab π dan kichik burchakka burish soat miliga qarama-qarshi bo‘lsa, bu $\vec{a}$, $\vec{b}$, $\vec{c}$ uchlik *o‘ng uchlik,* aks holda *chap uchlik* deyiladi. Chap va o‘ng uchlikni tashkil etadigan uchlik *tartiblangan uchlik* deb yuritiladi.

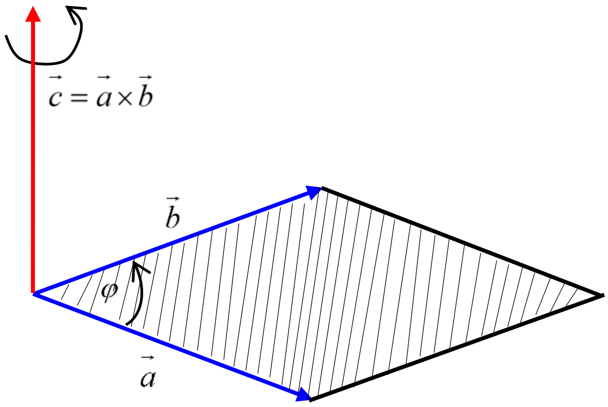
Biz o‘ng uchlikdan foydalanamiz.

$\vec{a}$ va $\vec{b}$ vektorlarning *vektor ko‘paytmasi* deb quyidagi shartlarni qanoatlantiradigan $\vec{c}$ vektorga aytiladi.

1) $\vec{c}$ vektor $\vec{a}$ va $\vec{b}$ vektorlarga perpendikulyar (ortogonal);

2) $|\vec{c}|=|\vec{a}|\cdot|\vec{b}|\cdot\sin(\vec{a},\vec{b})$; (7.1)

3) $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlarning tartiblangan uchligi o‘ng uchlikni tashkil etadi (3-chizma).



3-chizma.

(Bu ta’rifda $\vec{a}\neq 0$, $\vec{b}\neq 0$ deb faraz qilinadi) $\vec{a}$ va $\vec{b}$ vektorlarning vektor ko‘paytmasi $\vec{a}\times\vec{b}$ yoki $\left[\vec{a},\vec{b}\right]$ ko‘rinishida yoziladi. Agar $\vec{a}$ va $\vec{b}$ vektorlar kollinear bo‘lmasa, u holda $\left|\vec{c}\right|=\left|\vec{a}\times\vec{b}\right|$ son $\vec{a}$ va $\vec{b}$ vektorlarga ysalgan parallelogrammning S yuziga teng bo‘ladi. Shunday qilib, $S=\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot sin\left(\vec{a},\vec{b}\right)=\left|\vec{a}\times\vec{b}\right|$.

Agar $\vec{a}$ va $\vec{b}$ vektorlar kollinear bo‘lsa, u holda $\vec{a}\times\vec{b}=0$, chunki $\varphi=\left(\vec{a},\vec{b}\right)=0$ yoki $\varphi=\pi$ da $sin\left(\vec{a},\vec{b}\right)=0$.

1-misol

Agar $\left|\vec{a}\right|=8$, $\left|\vec{b}\right|=15$, $\vec{a}\cdot\vec{b}=96$ bo‘lsa, $\left|\vec{a}\times\vec{b}\right|$ ni hisoblang.

► $\vec{a}$ va $\vec{b}$ vektorlarning vektor ko‘paytmasi uzunligi, shu vektorlar uzunliklari ko‘paytmasi bilan ular orasidagi burchak sinusi ko‘paytmasiga teng. $\vec{a}$ va $\vec{b}$ vektorlarning skalyar ko‘paytmasi ga asosan:

$$\vec{a}\cdot\vec{b}=\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cos\left(\vec{a},\wedge\vec{b}\right)$$

Bundan

$$\cos\left(\vec{a},\wedge\vec{b}\right)=\frac{\vec{a}\cdot\vec{b}}{\left|\vec{a}\right|\cdot\left|\vec{b}\right|}=\frac{96}{8\cdot 15}=\frac{4}{5}$$

U holda

$$\sin\left(\vec{a},\wedge\vec{b}\right)=\sqrt{1-\cos^2\left(\vec{a},\wedge\vec{b}\right)}=\sqrt{1-\left(\frac{4}{5}\right)^2}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$

Demak,

$$\left|\vec{a}\times\vec{b}\right|=\left|\vec{a}\right|\cdot\left|\vec{b}\right|\sin\left(\vec{a},\wedge\vec{b}\right)=8\cdot 15\cdot\frac{3}{5}=72$$ ◄

Vektor ko‘paytma quyidagi qonunlarga bo‘ysunadi:

1. Vektor ko‘paytmada ko‘paytuvchilar o‘rnini almashtirilsa, uning ishorasi o‘zgaradi, ya’ni

$$\vec{a}\times\vec{b}=-(\vec{b}\times\vec{a})$$

2. Vektor ko‘paytma skalyar ko‘paytuvchiga nisbatan gruppalash qonuniga bo‘ysunadi, ya’ni

$$\left(\lambda\cdot\vec{a}\right)\times\vec{b}=\vec{a}\times\left(\lambda\cdot\vec{b}\right)=\lambda\cdot\left(\vec{a}\times\vec{b}\right)$$

3. $\vec{a}$ va $\vec{b}$ vektorlar yig‘indisi bilan $\vec{c}$ vektorning vektor ko‘paytmasi taqsimot qonuniga bo‘ysunadi, ya’ni

$$\left(\vec{a}+\vec{b}\right)\times\vec{c}=\vec{a}\times\vec{c}+\vec{b}\times\vec{c}$$

Endi vektor ko‘paytmaning koordinatalar orqali yozilishini ko‘rib o‘tamiz. Avvalo koordinata o‘qlarning $\vec{i},\vec{j},\vec{k}$ ortlar uchun quyidagi munosabatlar o‘rinli bo‘lishini eslatib o‘tamiz:

$$\begin{aligned}&\vec{i}\times\vec{i}=0\,, \quad \vec{i}\times\vec{j}=\vec{k}\,, \quad \vec{i}\times\vec{k}=-\vec{j}\,,\\&\vec{j}\times\vec{j}=0, \quad \vec{j}\times\vec{k}=\vec{i}\,, \quad \vec{j}\times\vec{i}=-\vec{k}\,,\\&\vec{k}\times\vec{k}=0\,, \quad \vec{k}\times\vec{i}=\vec{j}\,, \quad \vec{k}\times\vec{j}=-\vec{i}\,.\end{aligned} \tag{7.2}$$

Buni qisqacha quyidagi sxema orqali ham berish mumkin.

$$\left.\begin{array}{l}\overrightarrow{\vec{i}\times\vec{j}\times\vec{k}\times\vec{i}\times\vec{j}} \;\to +\\ \overleftarrow{\vec{i}\times\vec{j}\times\vec{k}\times\vec{i}\times\vec{j}} \;\leftarrow -\end{array}\right\} \tag{7.3}$$

$\vec{a}$ va $\vec{b}$ vektorlar Dekart koordinatalar sistemasida mos ravishda $\vec{a}\left\{a_x;a_y;a_z\right\}$ va $\vec{b}\left\{b_x;b_y;b_z\right\}$ koordinatalarga ega bo‘lsin, ya’ni

$$\vec{a}\left\{a_x;a_y;a_z\right\}=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}\,,\;\;\vec{b}\left\{b_x;b_y;b_z\right\}=b_x\vec{i}+b_y\vec{j}+b_z\vec{k}$$

$\vec{a}\times\vec{b}$ ko‘paytma uchun formulani (7.2) ni hamda vektor ko‘paytmaning xossalarini e’tiborga olib topamiz:

$$\vec{a}\times\vec{b}=a_xb_x\cdot\left(\vec{i}\times\vec{i}\right)+a_yb_x\cdot\left(\vec{j}\times\vec{i}\right)+a_zb_x\cdot\left(\vec{k}\times\vec{i}\right)+\;+a_xb_y\cdot\left(\vec{i}\times\vec{j}\right)+a_yb_y\cdot\left(\vec{j}\times\vec{j}\right)+a_zb_y\cdot\left(\vec{k}\times\vec{j}\right)+$$

$$+a_xb_z\cdot\left(\vec{i}\times\vec{k}\right)+a_yb_z\cdot\left(\vec{j}\times\vec{k}\right)+a_zb_z\cdot\left(\vec{k}\times\vec{k}\right).$$

yoki

$$\vec{a}\times\vec{b}=-a_yb_x\cdot\vec{k}+a_zb_x\cdot\vec{j}+a_xb_y\cdot\vec{k}-a_zb_y\cdot\vec{i}-a_xb_z\cdot\vec{j}+a_yb_z\cdot\vec{i}$$

Bir xil ortlarga ega bo‘lgan qo‘shiluvchilarni gruppalab yozamiz:

$$\vec{a}\times\vec{b}=\left(a_yb_z-a_zb_y\right)\cdot\vec{i}+\left(a_zb_x-a_xb_z\right)\cdot\vec{j}+\left(a_xb_y-a_yb_x\right)\cdot\vec{k}$$

Buni yana ushbu ko‘rinishda yozish mumkin:

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix} \tag{7.4}$$

Bu formuladan quyidagi ikki tasdiq kelib chiqadi.

1. (ikki vektorning kolleniar bo‘lish sharti). $\vec{a}$ va $\vec{b}$ vektorlar kolleniar bo‘lishi uchun $\vec{a}\times\vec{b}=0$ bo‘lishi zarur va etarli.
2. (uchburchak yuzining formulasi). $\vec{a}$ va $\vec{b}$ vektorlarga uchburchak yasalgan bo‘lsin, u holda bu uchburchakning yuzi:

$$S=\frac{1}{2}\left|\vec{a}\times\vec{b}\right|=\frac{1}{2}mod\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix} \tag{7.5}$$

(7.1) va (7.5) formulalar vektor ko‘paytmaning geometrik tatbiqlari hisoblanadi.

2-misol

Berilgan $\vec{a}\{2;0;3\}=2\vec{i}+3\vec{k}$ va $\vec{b}\{0;-4;1\}=-4\vec{j}+\vec{k}$ vektorlardan tuzilgan parallelogramning yuzini hisoblang.

► (7.1) ga binoan, $\left|\vec{a}\times\vec{b}\right|=\left|\vec{a}\right|\cdot\left|\vec{b}\right|\sin\left(\vec{a},\wedge\vec{b}\right)$. Vektor ko‘paytma xossalari va (7.2)ga asosan esa, $\vec{a}\times\vec{b}=\left(2\vec{i}+3\vec{k}\right)\times\left(-4\vec{j}+\vec{k}\right)=12\vec{i}-2\vec{j}-8\vec{k}$ bo‘ladi. Demak, parallelogramm yuzi $S=\left|\vec{a}\times\vec{b}\right|=\sqrt{12^2+(-2)^2+(-8)^2}=\sqrt{212}=2\sqrt{53}\,(kv.b.)$ ◄

Quyida aralash ko‘paytmaning fizik tatbiqiga bir masala ko‘ramiz:

3-misol

Agar $N(1,2,3)$ nuqtaga $\vec{F}=\vec{e}_1-2\vec{e}_2+4\vec{e}_3$ kuch qo‘yilgan bo‘lsa bu kuchning $M(3,\ 2,-1)$ nuqtaga nisbatan momenti topilsin.

► $\overrightarrow{MN}$ vektorni aniqlaymiz: $\overrightarrow{MN}=\{1-3,2-2,\ 3-(-1)\}$, $\overrightarrow{MN}=\{-2,\ 0,4\}$. N nuqtaga qo‘yilgan $\vec{F}$ kuchning momenti

$$m_N\left(\vec{F}\right)=\overrightarrow{MN}\times\vec{F}=\begin{vmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ \vec{F}_x & \vec{F}_y & \vec{F}_z\\ \left(\overrightarrow{MN}\right)_x & \left(\overrightarrow{MN}\right)_y & \left(\overrightarrow{MN}\right)_z\end{vmatrix}$$

formula bilan topiladi. Bu formulaga asosan quyidagini topamiz:

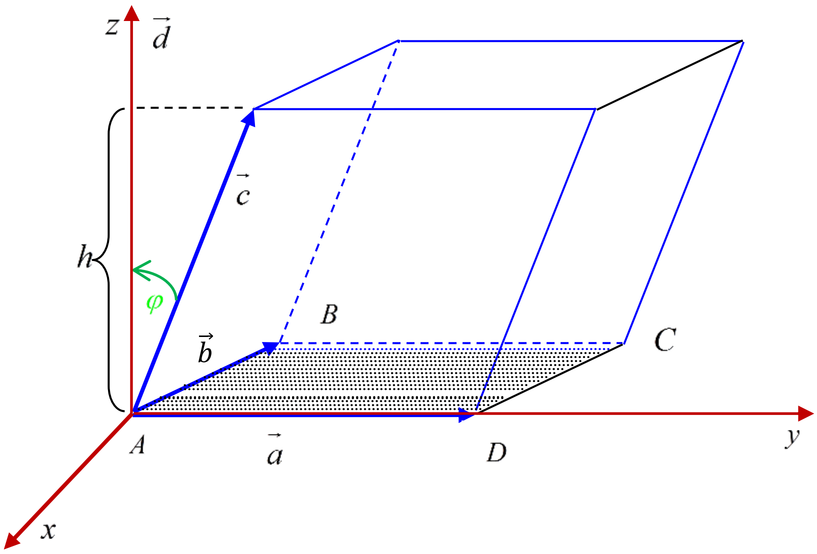
$$m_N\left(\vec{F}\right)=\overrightarrow{MN}\times\vec{F}=\begin{vmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ 1 & -2 & 4\\ -2 & 0 & 4\end{vmatrix}=-8\vec{e}_1-12\vec{e}_2-4\vec{e}_3 .$$ ◄

2.3.2 Vektorlarning aralash ko'paytmasi

$\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlar tartiblangan uchligining aralash ko'paytmasi deb, $\vec{a}\times\vec{b}$ vektor bilan $\vec{c}$ vektorning skalyar ko'paytmasiga teng songa aytiladi va $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$ yoki $\left[\vec{a},\vec{b}\right]\cdot\vec{c}$ kabi belgilanadi

Aralash ko'paytmaning moduli nuqtai nazardan ma'nosini tekshiramiz. $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlar komplanar bo'lmagan vektorlar bo'lsin. $\vec{a}\times\vec{b}=\vec{d}$ deb belgilasak, $\vec{d}$ vektor moduli $\vec{a}$ va $\vec{b}$ vektorlardan yasalgan parallelogram yuziga teng (4-chizma) $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=\vec{d}\cdot\vec{c}$ bo'lgani uchun skalyar ko'paytma ta'rifiga ko'ra

$$\vec{d}\cdot\vec{c}=\left|\vec{d}\right|\cdot Pr_{\vec{d}}\,\vec{c}$$



4-chizma.

Ammo $Pr_{\vec{d}}\,\vec{c}=h$ miqdorning moduli, ya'ni $|h|$ son $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlarga yasalgan parallelepipedning balandligini anglatadi.

Aralash ko'paytmaning absolyut qiymati shu $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlarga yasalgan parallelepiped hajmiga teng, ya'ni

$$V_{parallelepiped}=\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right|. \qquad (7.6)$$

Aralash ko'paytmaning ba'zi xossalarini keltiramiz:

1) Ko'paytmada ikki vektorning o'rinlari almashtirilsa, aralash ko'paytmaning ishorasi teskariga almashadi, ya'ni quyidagi tengliklar o'rinli:

$$(\vec{a}\times\vec{b})\cdot\vec{c}=-(\vec{b}\times\vec{a})\cdot\vec{c},$$
$$(\vec{a}\times\vec{b})\cdot\vec{c}=-(\vec{a}\times\vec{c})\cdot\vec{b},$$
$$(\vec{a}\times\vec{b})\cdot\vec{c}=-(\vec{c}\times\vec{b})\cdot\vec{a}.$$

2) $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlarning o‘rinlari “doiraviy shaklda” almashtirilsa, aralash ko‘paytma o‘z ishorasini o‘zgartirmaydi, ya’ni ushbu tengliklar o‘rinli:

$$(\vec{a}\times\vec{b})\cdot\vec{c}=(\vec{b}\times\vec{c})\cdot\vec{a}=(\vec{c}\times\vec{a})\cdot\vec{b}.$$

3) Agar $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlardan istalgan ikkitasi bir-biriga teng yoki parallel (kollinear) bo‘lsa, ularning aralash ko‘paytmasi nolga teng bo‘ladi.
4) Agar $\vec{a}$, $\vec{b}$ $\vec{\ }$ vektorlar o‘zaro komplanar vektorlar bo‘lsa, ularning aralash ko‘paytmasi nolga teng.

Endi aralash ko‘paytmani $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlarning koordinatalari orqali ifodalashga o‘tamiz. Dekart koordinatalar sistemasiga nisbatan $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlarning yoyilmasi berilgan bo‘lsin:

$$\vec{a}=\{x_1,y_1,z_1\}\Leftrightarrow\vec{a}=x_1\vec{i}+y_1\vec{j}+z_1\vec{k}$$
$$\vec{b}=\{x_2,y_2,z_2\}\Leftrightarrow\vec{b}=x_2\vec{i}+y_2\vec{j}+z_2\vec{k}$$
$$\vec{c}=\{x_3,y_3,z_3\}\Leftrightarrow\vec{c}=x_3\vec{i}+y_3\vec{j}+z_3\vec{k}$$

U holda

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i}_1 & \vec{j} & \vec{k}\\ x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\end{vmatrix}=\begin{vmatrix}y_1 & z_1\\ y_2 & z_2\end{vmatrix}\vec{i}+\begin{vmatrix}z_1 & x_1\\ z_2 & x_2\end{vmatrix}\vec{j}+\begin{vmatrix}x_1 & y_1\\ x_2 & y_2\end{vmatrix}\vec{k}.$$

Shuning uchun

$$(\vec{a}\times\vec{b})\cdot\vec{c}=x_3\begin{vmatrix}y_1 & z_1\\ y_2 & z_2\end{vmatrix}+y_3\begin{vmatrix}z_1 & x_1\\ z_2 & x_2\end{vmatrix}+z_3\begin{vmatrix}x_1 & y_1\\ x_2 & y_2\end{vmatrix}=\begin{vmatrix}x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{vmatrix}.$$

Shunday qilib, uch vektor aralash ko‘paytmasining uchinchi tartibli determinant orqali ifodasi ushbu ko‘rinishda bo‘ladi:

$$\Delta=(\vec{a}\times\vec{b})\cdot\vec{c}=\begin{vmatrix}x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{vmatrix}. \qquad (7.7)$$

Formuladan kelib chiqadigan ba’zi natijalarni keltiramiz.

1-Natija. $\vec{a}$, $\vec{b}$, $\vec{c}$ *vektorlar komplanar bo'lishi uchun*

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0 \qquad (7.8)$$

tenglikning bajarilishi zarur va yetarli.

4-misol

Berilgan $\vec{a} = \{2; -1; 3\}$, $\vec{b} = \{3; 0; 2\}$, $\vec{c} = \{1; -1; 4\}$ vektorlarni chiziqli erklilikka tekshiring.

► Agar uch vektor komplanar bo'lsa, ular chiziqli bog'liq bo'ladi. Chunki tekislikda har qanday uch vektor chiziqli bog'liqdir. Berilgan vektorlarni komplanarlikka tekshirish kifoya.

$$\begin{vmatrix} 2 & -1 & 3 \\ 3 & 0 & 2 \\ 1 & -1 & 4 \end{vmatrix} = 0 - 2 - 9 - 0 + 4 + 12 = 5 \neq 0 .$$

Demak, berilgan vektorlar chiziqli erkli ekan.◄

2-Natija. *Agar* $\vec{a} = \{x_1, y_1, z_1\}$, $\vec{b} = \{x_2, y_2, z_2\}$, $\vec{c} = \{x_3, y_3, z_3\}$ *bo'lib, bu vektorlar komplanar bo'lmasa, u holda ularga qurilgan parallelepiped hajmi* $V = \pm\Delta$ *formula o'rinli. Unda musbat ishora* $\vec{a}$, $\vec{b}$, $\vec{c}$ *o'ng uchlikni, manfiy ishora shu* $\vec{a}$, $\vec{b}$, $\vec{c}$ *lar chap uchlikni tashkil etganda olinadi.*

5-misol

Berilgan $\vec{a} = 4\vec{i} + 3\vec{j} - 2\vec{k}$, $\vec{b} = -2\vec{i} - \vec{j} + 2\vec{k}$, $\vec{c} = 2\vec{i} + \vec{j} + x\vec{k}$ vektorlardan tuzilgan piramidaning hajmi 8 ga teng bo'lsa, x ni toping.

► (7.7) ko'ra, aralash ko'paytmani topamiz.

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} 4 & 3 & -2 \\ -2 & -1 & 2 \\ 2 & 1 & x \end{vmatrix} = -4x + 12 + 4 - 4 - 8 + 6x = 2x + 4 .$$

$V_{pir.} = \frac{1}{6} V_{par-d}$ bo'lgani uchun va (7.6) dan, $V_{pir.} = \frac{1}{6} |2x + 4| = 8$, $|2x + 4| = 48$.

U holda, $x_1 = -26$ va $x_2 = 22$.◄

Auditoriya topshiriqlari

1. Uchlari $A(1;2;0)$, $B(3;0;-3)$, $C(5;2;6)$ nuqtalarda bo'lgan uchburchak yuzini hisoblang.

Javob: $S_{\Delta ABC}=0.5\cdot\left|\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\right|=14$ kv. birlik.

2. $\overrightarrow{AB}=-3\vec{i}-22\vec{j}+6\vec{k}$; $\overrightarrow{BC}=-2\vec{i}+4\vec{j}+4\vec{k}$ vektorlar ΔABC ning tomonlari. $\overrightarrow{AD}$ balandlikning uzunligini hisoblang.

Javob: $\left|\overrightarrow{AD}\right|=\dfrac{2S_{\Delta ABC}}{\left|\overrightarrow{BC}\right|}=\dfrac{8\sqrt{5}}{3}$.

3. $\vec{a}$, $\vec{b}$ va $\vec{c}$ vektorlar koordinatalari bilan berilgan:

$$\vec{a}=2\vec{i}-\vec{j}-\vec{k}\ ;\ \vec{b}=\vec{i}+3\vec{j}-\vec{k};\vec{c}=3\vec{i}-4\vec{j}+7\vec{k}\,.$$

Bu vektorlarning aralash ko'paytmasini toping.

Javob: $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=33$.

4. Ushbu $\vec{a}=2\vec{i}-\vec{j}+2\vec{k}$; $\vec{b}=\vec{i}+2\vec{j}-3\vec{k}$; $\vec{c}=3\vec{i}-4\vec{j}+7\vec{k}$ vektorlarning komplanarligini isbotlang.

5. Uchlari $A(1;2;3)$, $B(2;4;1)$, $C(7;6;3)$ va $D(2;-3;-1)$ nuqtalarda bo'lgan piramida berilgan. Shu piramida uchun quyidagilarni: a) AB, AC, AD qirralarning uzunliklarini; b) ABC yoqning yuzini; d) piramidaning hajmini toping.

Javob:

$a)\ \left|\overrightarrow{AB}\right|=\sqrt{17},\ \left|\overrightarrow{AC}\right|=2\sqrt{13},\ \left|\overrightarrow{AD}\right|=5\sqrt{2}$;

$b) S_{\Delta ABC}=\dfrac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=14$ kv. birlik;

$d)\ V_{pir}=30$ kub birlik.

6. Agar tekislikda $\vec{a}$ va $\vec{b}$ vektorlar nokollinear bo'lsa α ning qanday qiymatida $\vec{p}=\alpha\vec{a}+2\vec{b}$ va $\vec{q}=3\vec{a}-\vec{b}$ vektorlar kollinear bo'ladi.

Javob: $\alpha=-6$

7. Agar $|\vec{a}|=3$, $\left|\vec{b}\right|=4$ va $(\vec{a}\,\hat{}\,\vec{b})=\dfrac{\pi}{3}$ bo'lsa $\vec{p}=3\vec{a}-5\vec{b},$ $\vec{q}=\vec{a}+7\vec{b}$ vektorlardan tuzilgan uchburchak yuzini toping.

Javob: $S_{\Delta}=\dfrac{1}{2}|\vec{p}\times\vec{q}|=78\sqrt{3}$

8. $C(-1;\,4;-2)$ nuqtaga qo'yilgan uchta $\vec{F}=\{2;-1;-2\}$, $\vec{Q}=\{3;\,2;-1\}$ va $\vec{P}=\{-4;\,1;\,3\}$ kuchlar berilgan. Bu kuchlar teng ta'sir etuvchisining $A(2;\,3;-1)$ nuqtaga nisbatan momentining yo'naltiruvchi kosinuslarini toping.

Javob: $cos\alpha = \frac{1}{\sqrt{66}}$, $\cos\beta = -\frac{4}{\sqrt{66}}$, $\cos\gamma = -\frac{7}{\sqrt{66}}$

Mustaqil yechish uchun testlar

1. Berilgan $\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}$, $\vec{b} = -\vec{i} + 3\vec{j} + 2\vec{k}$ vektorlarning vektor ko'paytmasi $\vec{a} \times \vec{b}$ ni toping

A) $\{9;-3;9\}$; B) $\{6;3;-9\}$; C) $\{-9;3;-5\}$; D) $\{9;3;6\}$

2. Agar $|\vec{a}| = 5$, $|\vec{b}| = 8$ va $(\vec{a}\,\hat{}\,\vec{b}) = \frac{\pi}{6}$ bo'lsa, $\left|\left(2\vec{a} + 3\vec{b}\right) \times \left(\vec{a} - 2\vec{b}\right)\right|$ ni hisoblang

A) $75\sqrt{3}$; B) 105; C) 140; D) $60\sqrt{3}$

3. Agar $\bar{a}(1;\ 2;-3)$, $\bar{b}(-2;1;-1)$ bo'lsa, $(\bar{a} - 2\bar{b}) \times (2\bar{a} - \bar{b})$ vektor ko'paytmani toping

A) $(3;-21;15)$; B) $(-5;-35;-25)$; C) $(3;21;15)$; D) $(5;-35;25)$

4. $\vec{a}(2;1;6)$, $\vec{b}(1;-2;-1)$ va $\vec{c}(2;-4;-2)$ vektorlarning aralash ko'paytmasini hisoblang

A) 0; B) -4; C) 6; D) 4

5. Agar $\vec{a} = \{x;-1;2\}$, $\vec{b} = \{1;x;-3\}$, $\vec{c} = \{1;-3;5\}$ vektorlar chiziqli bog'liq bo'lsa, x ning qiymatini toping

A) $x_1 = -2,\ x_2 = 0{,}2$ B) $x_1 = 2,\ x_2 = 0{,}2$

C) $x_1 = -2,\ x_2 = -0{,}2$ D) $x_1 = 2,\ x_2 = -0{,}2$

Shaxsiy uy topshiriqlari

1. Agar $|\vec{a}| = 13$, $|\vec{b}| = 19$, va $|\vec{a} + \vec{b}| = 24$ bo'lsa, $|\vec{a} - \vec{b}|$ ni hisoblang.

Javob: $|\vec{a} - \vec{b}| = 22$.

2. Agar ΔABC da $\overrightarrow{AB} = \vec{m}$, $\overrightarrow{AC} = \vec{n}$ ekanligi ma'lum bo'lsa, quyidagi vektorlarni yasang:

1) $\frac{\vec{m} + \vec{n}}{2}$, 2) $\frac{\vec{m} - \vec{n}}{2}$, 3) $\frac{\vec{n} - \vec{m}}{2}$, 4) $-\frac{\vec{m} + \vec{n}}{2}$

3. $\vec{a}$, $\vec{b}$ vektorlarga yasalgan parallelogrammdan foydalanib quyidagi ayniyatlarning to'g'riligini chizmada tekshiring:

1) $\left(\vec{a} + \vec{b}\right) + \left(\vec{a} - \vec{b}\right) = 2\vec{a}$, 2) $\vec{a} + \left(\vec{b} - \vec{a}\right) = \vec{b}$ 3) $\frac{\vec{a} - \vec{b}}{2} + \vec{b} = \frac{\vec{a} + \vec{b}}{2}$;

4. Teng yonli $ABCD$ trapesiyaning pastki asosi $\overrightarrow{AB} = \vec{a}$, yon tomoni $\overrightarrow{AD} = \vec{b}$ va ular orasidagi burchagi $\alpha = \frac{\pi}{3}$ berilgan. Trapesiyaning qolgan tomonlari va diagonallarini tashkil etuvchi vektorlarni $\vec{a}$ va $\vec{b}$ vektorlar orqali ifodalang

Javob: $\overrightarrow{BC}=-\frac{b}{a}\vec{a}+\vec{b}$; $\overrightarrow{CD}=\frac{b-a}{a}\vec{a}$; $\overrightarrow{AC}=\frac{a-b}{a}\vec{a}+\vec{b}$; $\overrightarrow{BD}=-\vec{a}+\vec{b}$; bu yerda a, b mos ravshda $\vec{a}$, $\vec{b}$ vektorlarning uzunliklarini bildiradi.

5.Koordinatalar boshidan $M(12;-3;4)$ nuqtagacha bo‘lgan masofani hisoblang.

$\vec{r}(0;2;-3)$ radius vektorning ortlar bo‘yicha yoyilmasini yozing va modulini hisoblang.

Javob: $\vec{r}=2\vec{j}-3\vec{k}$, $|\vec{r}|=\sqrt{13}.$

6. $M(-2;\,1;\,3)$ va $N(0;-1;\,2)$ nuqtalar orasidagi masofani toping.

Javob: 3.

7. $\vec{a}\{3,\ 2,\ 7\}$ va $\vec{b}\{4,\ 1,\ -5\}$ vektorlarning yig‘indisi va ayirmasini ort vektorlar yordamida yozing.

Javob:
$$\vec{a}+\vec{b}=7\vec{i}+3\vec{j}+2\vec{k}$$
$$\vec{a}-\vec{b}=-\vec{i}+\vec{j}+12\vec{k}$$

8.Uchlari $A(5;\,2;\,6)$, $B(6;\,4;\,4)$, $C(4;\,3;\,2)$ va $D(3;\,1;\,4)$ nuqtalarda bo‘lgan to‘rtburchakning kvadrat ekanligini tekshiring.

9. α va β larning qanday qiymatlarida $\vec{a}=2\vec{i}+\alpha\vec{j}+\vec{k}$ va $\vec{b}=3\vec{i}-6\vec{j}+\beta\vec{k}$ vektorlar kollinear bo‘ladi?

Javob: $\alpha=-4$; $\beta=\frac{3}{2}$.

10.Uchlari $A(2;\,1;\,-4)$, $B(1;\,3;\,5)$, $C(7;\,2;\,3)$ va $D(8;\,0;\,-6)$ nuqtalarda bo‘lgan to‘rtburchakning parallelogramm ekanligini isbotlang va parallelogramm tomonlari uzunliklarini toping.

Javob. $\overrightarrow{AB}=\overrightarrow{DC}$ bo‘lgani uchun parallelogramdir.

$|\overrightarrow{AB}|=\sqrt{86}\approx 9,3$; $|\overrightarrow{DC}|=\sqrt{41}\approx 6,4.$

11. Uchlari $A(-1;\,2;\,3)$, $B(2;\,-1;\,1)$, $C(1;\,-3;\,-1)$ va $D(-5;\,3;\,3)$ nuqtalarda bo‘lgan to‘rtburchakning trapesiya ekanligini isbotlang

Ko‘rsatma . $\overrightarrow{AB}$ va $\overrightarrow{CD}$ vektorlarning kollinear. $\overrightarrow{AD}$ va $\overrightarrow{BC}$ vektorlarning kollinear emasligini tekshirish zarur.

12.Boshlang‘ich nuqtasi $M(-1;\,3;\,2)$ va oxirgi nuqtasi $N(0;\,1;\,4)$ bo‘lgan $\overrightarrow{MN}$ vektorning yo‘naltiruvchi kosinuslarini toping.

Javob. $\cos\alpha=\frac{1}{3}$; $\cos\alpha=-\frac{2}{3}$; $\cos\alpha=\frac{2}{3}$.

13. $\vec{a}$ vektor Ox o‘qi bilan $\alpha=45^\circ$, Oy o‘qi bilan $\beta=60^\circ$ burchak hosil qiladi. Agar $|\vec{a}|=6$ bo‘lsa, uning koordinatalari topilsin.

Javob. $\vec{a}\left\{3\sqrt{2};\,3;\,3\right\}$.

14. $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak $\varphi=\frac{\pi}{3}$ ga teng. $|\vec{a}|=4$; $|\vec{b}|=3$ bo‘lsa, $\vec{c}=3\vec{a}+2\vec{b}$ vektorning uzunligini toping.

Javob. $\left|\vec{c}\right| = 2\sqrt{63}$. .

15. $\vec{a}$ va $\vec{b}$ vektorlar koordinatalari bilan berilgan: $\vec{a} = 7\vec{i} + 2\vec{j} + 3\vec{k}$; $\vec{b} = 2\vec{i} + 2\vec{j} + 4\vec{k}$. Bu vektorlarning skalyar ko‘paytmasini toping.

16.Uchlari $A(-1;\ 5;\ 1)$, $B(1;\ 1;\ -2)$, $C(-3;\ 3;\ 2)$ nuqtalarda bo‘lgan uchburchak berilgan. AC tomonni davom ettirishdan hosil bo‘lgan tashqi burchakni aniqlang.

Javob. $\varphi = arccos\left(\frac{4}{9}\right)$.

III BOB ANALITIK GEOMETRIYA ASOSLARI

3.1 Tekislikda to‘g‘ri chiziq tenglamalari

To‘g‘ri burchakli Dekart koordinatalar sistemasi *Oxy* tekislikda har qanday to‘g‘ri chiziq *x* va *y* ga nisbatan birinchi darajali

$$Ax + By + C = 0 \qquad (1.1)$$

tenglama bilan beriladi, bu yerda *A*, *B*, *C* –haqiqiy sonlar, $A^2 + B^2 > 0$ va har qanday (1.1) tenglama to‘g‘ri chiziqni aniqlaydi.

(1.1) tenglama *to‘g‘ri chiziqning umumiy tenglamasi* deyiladi. To‘g‘ri chiziqqa perpendikulyar $\vec{n} = \{A; B\}$ vektor to‘g‘ri chiziqning *normal vektori* deyiladi.

Agar $B \neq 0$ bo‘lsa, (1.1)ni *y* ga nisbatan yechib,

$$y = kx + b \ (k = tg\alpha) \qquad (1.2)$$

ko‘rinishda ifodalash mumkin. (1.2) tenglama *to‘g‘ri chiziqning burchak koeffitsientli tenglamasi* deyiladi. α - to‘g‘ri chiziq bilan *Ox* o‘qining musbat yo‘nalishi orasidagi burchak, *k* - to‘g‘ri chiziqning burchak koeffitsienti, *b* - to‘g‘ri chiziqning *Oy* o‘qidan kesadigan kesmasi.

To‘g‘ri chiziqning yana quyidagi tenglamalari mavjud:

1. $M_0(x_0; y_0)$ nuqtadan o‘tuvchi va $\vec{n} = \{A; B\}$ normal vektorga ega to‘g‘ri chiziq tenglamasi:

$$A(x - x_0) + B(y - y_0) = 0 \qquad (1.3)$$

2. $M_0(x_0; y_0)$ nuqtadan o‘tuvchi va *k* - burchak koeffitsientli to‘g‘ri chiziq tenglamasi:

$$y - y_0 = k(x - x_0) \qquad (1.4)$$

3. To‘g‘ri chiziqning parametrik tenglamasi:

$$\begin{cases} x = x_0 + mt \\ y = y_0 + nt \end{cases} \qquad (1.5)$$

Bu yerda, $\vec{s}(m; n)$ - to‘g‘ri chiziqning *yo‘naltiruvchi vektori*.

4. To‘g‘ri chiziqning kanonik tenglamasi:

$$\frac{x - x_0}{m} = \frac{y - y_0}{n} \qquad (1.6)$$

5. To‘g‘ri chiziqning “kesma”lardagi tenglamasi:

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad (1.7)$$

Bu yerda *a* va *b* to‘g‘ri chiziqning mos ravishda *Ox* va *Oy* koordinata o‘qlaridan ajratgan kesmalari.

6. Ikki $M_1(x_1; y_1)$ va $M_2(x_2; y_2)$ nuqtalardan o‘tuvchi to‘g‘ri chiziq tenglamasi:

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1} \tag{1.8}$$

1-misol

Quyidagi $2x-3y+6=0$ tenglama bilan berilgan to‘g‘ri chiziqning burchak koeffitsientini va o‘qlardan ajratgan kesmalarini aniqlang.

► $2x-3y+6=0$ ni y ga nisbatan yechamiz: $y=\frac{2}{3}x+2$, $k=\frac{2}{3}$. Berilgan tenglamani quyidagicha almashtiramiz:

$$2x-3y=-6 \,|\, :(-6)$$

$$\frac{x}{-3}+\frac{y}{2}=1$$

Demak, $a=-3$, $b=2$, $k=\frac{2}{3}$.◄

2-misol

ABC uchburchakning uchlari $A(-3;1)$, $B(5;-3)$ va $C(7;5)$ berilgan.

CD balandlik va *AE* medianalari kesishgan nuqtasini toping.

► CD balandlik *AB* tomonga perpendikulyar bo‘lishi kerak. Avval (1.8) ni qo‘llab, *AB* tomon tenglamasini tuzamiz.

$$\frac{x+3}{5+3}=\frac{y-1}{-3-1},\ y=-\frac{1}{2}x-\frac{1}{2},\ k_1=-\frac{1}{2}.$$

CD balandlik tenglamasida $k_2=2$, u holda (1.2)ga ko‘ra, $y-5=2(x-7)$ yoki $y=2x-9$.

E nuqta $B(5;-3)$ va $C(7;5)$ nuqtalarning o‘rtasi bo‘lgani uchun

$$E\left(\frac{5+7}{2};\frac{-3+5}{2}\right)=E(6;1).$$

$A(-3;1)$ va $E(6;1)$ nuqtalardan o‘tuvchi *AE* mediana tenglamasi: $y=1$.

CD balandlik va *AE* medianalar tenglamalarini birgalikda yechamiz:

$$\begin{cases} y=2x-9 \\ y=1 \end{cases};\quad \begin{cases} x=5 \\ y=1 \end{cases}.$$

Demak, $M(5;1)$ - *CD* balandlik va *AE* medianalar kesishgan nuqta.◄

Tekislikda to‘g‘ri chiziqlarning o‘zaro joylashish holatlarini ko‘rib chiqamiz.

1. Agar tekislikda to‘g‘ri chiziqlar

$$A_1x+B_1y+C_1=0 \text{ va } A_2x+B_2y+C_2=0$$

umumiy tenglamalar bilan berilgan bo‘lsin. U holda ular orasidagi φ burchaklardan biri ularning $\vec{n}_1=\{A_1;\,B_1\}$ va $\vec{n}_2=\{A_2;\,B_2\}$ normallari orasidagi burchakga teng va quyidagi formula bilan hisoblanadi:

$$\cos\varphi=\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1|\cdot|\vec{n}_2|}=\frac{A_1A_2+B_1B_2}{\sqrt{A_1^2+B_1^2}\sqrt{A_2^2+B_2^2}} \tag{1.9}$$

To‘g‘ri chiziqlarning *perpendikulyarlik sharti*

$$A_1A_2+B_1B_2=0 \tag{1.10}$$

formula bilan aniqlanadi.

To‘g‘ri chiziqlarning *parellellik sharti*

$$\frac{A_1}{A_2}=\frac{B_1}{B_2}\neq\frac{C_1}{C_2} \tag{1.10}$$

formula bilan aniqlanadi.

Agar

$$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2} \tag{1.11}$$

tenglik bajarilsa, to‘g‘ri chiziqlar ustma-ust tushadi.

2. Tekislikda to‘g‘ri chiziqlar $y=k_1x+b_1$ va $y=k_2x+b_2$ burchak koeffitsientli tenglamalar bilan berilgan bo‘lsin. U holda ular orasidagi φ burchak

$$tg\varphi=\frac{k_2-k_1}{1+k_1k_2} \tag{1.12}$$

formula bilan hisoblanadi. Bu holda to‘g‘ri chiziqlar parallel bo‘lishi uchun $k_1=k_2$ tenglik bajarilishi va perpendikulyar bo‘lishi uchun $k_1k_2=-1$ shart bajarilishi zarur va yetarli.

$M_0(x_0;y_0)$ nuqtadan $Ax+By+C=0$ to‘g‘ri chiziqgacha bolgan d masofa quyidagi

$$d=\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}} \tag{1.13}$$

formula bilan hisoblanadi.

3-misol

Berilgan $x-2y+4=0$ to‘g‘ri chiziqqa nisbatan $M(5;\,2)$ nuqtaga simmetrik nuqtani toping.

►Avval $M(5;\,2)$ nuqtadan o‘tuvchi va $\vec{n}=\{1;-2\}$ normal vektorli $x-2y+4=0$ to‘g‘ri chiziqqa perpendikulyar to‘g‘ri chiziq tenglamasini tuzamiz. Bu holda $\vec{n}=\{1;-2\}$ izlanayotgan to‘g‘ri chiziqning yo‘naltiruvchi vektori bo‘ladi. (1.6)ga ko‘ra,

$$\frac{x-5}{1}=\frac{y-2}{-2},\ y=-2x+12.$$

Bu to‘g‘ri chiziqlar kesishish nuqtasini topamiz.

$$\begin{cases}x-2y+4=0\\ y=-2x+12\end{cases}\Rightarrow\begin{cases}x=4\\ y=4\end{cases}.$$

Topilgan $M_0(4;4)$ nuqta $M(5;\,2)$ nuqta va unga simmetrik $M'(x;\,y)$ nuqtalarning o‘rtasi bo‘lgani uchun

$$\frac{x+5}{2}=4, \qquad \frac{y+2}{2}=4$$

tenglik o‘rinli. Bundan, $x=3,\ y=6$. Demak, $M'(3;\,6)$.◄

4-misol

Kvadratning ikkita tomoni $5x-12y-65=0$ va $5x-12y+26=0$ to‘g‘ri chiziqlarda yotsa, kvadratning yuzini toping.

► Berilgan to‘g‘ri chiziqlar o‘zaro parallel bo‘lgani uchun ular kvadratning qarama-qarshi tomonlari bo‘lib, orasidagi masofa kvadrat tomonining uzunligiga teng. Buning uchun $5x-12y-65=0$ to‘g‘ri chiziqdan ixtiyoriy nuqta tanlanadi, masalan, $M_0(1;-5)$ va ikkinchi $5x-12y+26=0$ to‘g‘ri chiziqgacha masofa (1.13)ga asosan topiladi.

$$d=\frac{|5\cdot 1-12\cdot(-5)+26|}{\sqrt{5^2+(-12)^2}}=\frac{91}{13}=7\ .$$

Demak, $S_{kv}=49$.◄

Auditoriya topshiriqlari

1. $2x-5y+8=0$ tenglama bilan berilgan to‘g‘ri chiziqning burchak koeffitsientini va o‘qlardan ajratgan kesmalarini aniqlang.

2. $A(5;-3)$ nuqtadan o‘tuvchi va *a*) Ox o‘qiga; *b*) Oy o‘qiga; c) 1-chorak bissektrisasiga; d) $y=-2x+7$ to‘g‘ri chiziqga; e) $2x-5y+8=0$ to‘g‘ri chiziqga parallel to‘g‘ri chiziq tenglamalarini tuzing.

3. $A(-1;3)$ va $B(2;-5)$ nuqtalardan o‘tuvchi to‘g‘ri chiziq tenglamasini tuzing.

4. $\mathrm{A}(-2;5)$ nuqtadan o‘tib, $3x+5y-8=0$ to‘g‘ri chiziqga perpendikulyar to‘g‘ri chiziq tenglamasini tuzing.

5. Kvadratning bir uchi $A(-1;2)$ nuqtada, bir tomoni esa $4x-3y-15=0$ to‘g‘ri chiziqda yotadi. Kvadratning yuzini hisoblang.

6. $4x-3y-12=0$ to‘g‘ri chiziqqa parallel va undan $d=2$ masofada joylashgan to‘g‘ri chiziq tenglamasini tuzing.

7. Agar $M(4;\,2)$ nuqta to‘g‘ri chiziqning koordinatalar orasidagi kesmasining o‘rtasi ekani ma’lum bo‘lsa to‘g‘ri chiziq tenglamasini tuzing.

8. $A(1;-2)$, $B(5;4)$ va $\mathrm{C}(-2;0)$ nuqtalar uchburchakning uchlari bo‘lsa, uning bissektrisalari tenglamalarini tuzing.

9. To‘g‘ri chiziqning $A(3;-4)$ nuqtasi unga koordinata boshidan tushirilgan perpendikulyar asosi ekani ma’lum bo‘lsa, bu to‘g‘ri chiziq tenglamasini tuzing.

10. $5x-y+4=0$ va $3x+2y-1=0$ to‘g‘ri chiziqlar orasidagi burchakni toping.

Mustaqil yechish uchun testlar

1. $3x+5y-8=0$ to‘g‘ri chiziqning burchak koeffitsientini va Oy o‘qidan ajratgan kesmasini aniqlang

$A)\ \ k=\frac{3}{5};\ b=\frac{8}{5}\quad B)\ \ k=-\frac{3}{5};\ b=\frac{8}{5};\quad C)\ \ k=\frac{5}{3};\ b=\frac{8}{3};\quad D)\ \ k=\frac{5}{3};\ b=-\frac{8}{3}$

2. Berilgan $A(3;-4)$ va $B(1;-3)$ va nuqtalardan o‘tuvchi to‘g‘ri chiziqning umumiy tenglamasini toping

$A)\ \ \frac{x-3}{-2}=\frac{y+4}{1};\quad B)\ \ y=-\frac{1}{2}x-\frac{5}{2}\quad C)\ \ x+2y+5=0;\quad D)\ \ \begin{cases}x=3-2t\\ y=-4+t\end{cases}$

3. Berilgan $A(3;-4)$ va $B(1;-3)$ va nuqtalardan o‘tuvchi to‘g‘ri chiziqning parametrik tenglamasini toping

$A)\ \ \frac{x-3}{-2}=\frac{y+4}{1};\quad B)\ \ y=-\frac{1}{2}x-\frac{5}{2}\quad C)\ \ x+2y+5=0;\quad D)\ \ \begin{cases}x=3-2t\\ y=-4+t\end{cases}$

4. Quyidagilardan qaysi biri $M(1;-3)$ nuqtadan o‘tib, $\vec{s}=\{-3;5\}$ vektorga parallel bo‘lgan to‘g‘ri chiziq bo‘ladi?

$A)\ \ \frac{x+3}{-1}=\frac{y-5}{3};\quad B)\ \ y=-\frac{1}{2}x-\frac{5}{2}\quad C)\ \ \frac{x-1}{3}=\frac{y+3}{-5};\quad D)\ \ \begin{cases}x=1-3t\\ y=-3-5t\end{cases}$

5. Trapetsiya asoslarining tenglamalari berilgan: $3x-4y-15=0,\ 3x-4y-35=0$. Trapetsiyaning balandligini aniqlang

$A)\ \ \frac{3}{5};\quad B)\ \ 3\quad C)\ \ 4\quad D)\ \ 5$

3.2 Fazoda tekislik tenglamalari

To‘g‘ri burchakli Dekart koordinatalar sistemasida ixtiyoriy tekislik

$$Ax+By+Cz+D=0 \tag{2.1}$$

tenglama bilan beriladi, bu yerda *A*, *B*, *C*, *D* – ma’lum sonlar, $A^2+B^2+C^2>0$ va (2.1) ko‘rinishdagi har qanday tenglama biror tekislikni aniqlaydi. (2.1) tenglama *tekislikning umumiy tenglamasi* deb ataladi. (2.1) tenglama bilan berilgan tekislikka perpendikulyar $\vec{n}(A;B;C)$ vektor tekislikning *normal vektori*(yoki *normali*) deyiladi.

Tekislikning bir nechta berilish usullari mavjud.

1. Berilgan $M_0(x_0,y_0,z_0)$ nuqtadan o‘tuvchi va $\vec{n}(A;B;C)$ normal vektorga ega tekislik tenglamasi:

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0 \tag{2.2}$$

2. Tekislikning o‘qlardan ajratgan kesmalar bo‘yicha tenglamasi:

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \tag{2.3}$$

Agar (2.1)da $D\neq 0$ bo‘lsa, $-D$ga bo‘lish orqali (2.3) tenglama hosil qilinadi va bu yerda $a,\ b,\ c$ tekislikning mos ravishda Ox, Oy, Oz o‘qlardan ajratgan kesmalaridir.

3. Uch nuqtadan o‘tuvchi tekislik tenglamasi. Agar tekislik bir to‘g‘ri chiziqda yotmaydigan $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$ va $M_3(x_3, y_3, z_3)$ nuqtalardan o‘tsa, uning tenglamasi quyidagicha yoziladi:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0 \qquad (2.4)$$

Determinantni 1-satr elementlari bo‘yicha yoyish orqali (2.2) formulani hosil qilish mumkin.

Tekisliklar orasidagi φ burchak deganda ular hosil qiladigan ikki yoqli burchaklardan biri tushuniladi.

(P_1): $A_1x + B_1y + C_1z + D_1 = 0$ va (P_2): $A_2x + B_2y + C_{21}z + D_2 = 0$ tekisliklar fazoda har qanday joylashganda ham ular orasidagi burchaklardan biri ularning $\vec{n}_1 = \{A_1; B_1; C_1\}$ va $\vec{n}_2 = \{A_2; B_2; C_2\}$ normallari orasidagi burchakka teng(1-shakl). Shuning uchun tekisliklar orasidagi burchak quyidagi formula yordamida hisoblanadi:

$$cos\varphi = \cos\left(\widehat{\vec{n}_1, \vec{n}_2}\right) = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \sqrt{A_2^2 + B_2^2 + C_2^2}} \qquad (2.5)$$

Tekisliklarning *perpendikulyarlik sharti*

$$A_1A_2 + B_1B_2 + C_1C_2 \qquad (2.6)$$

formula bilan aniqlanadi.

Tekisliklarning *parellellik sharti*

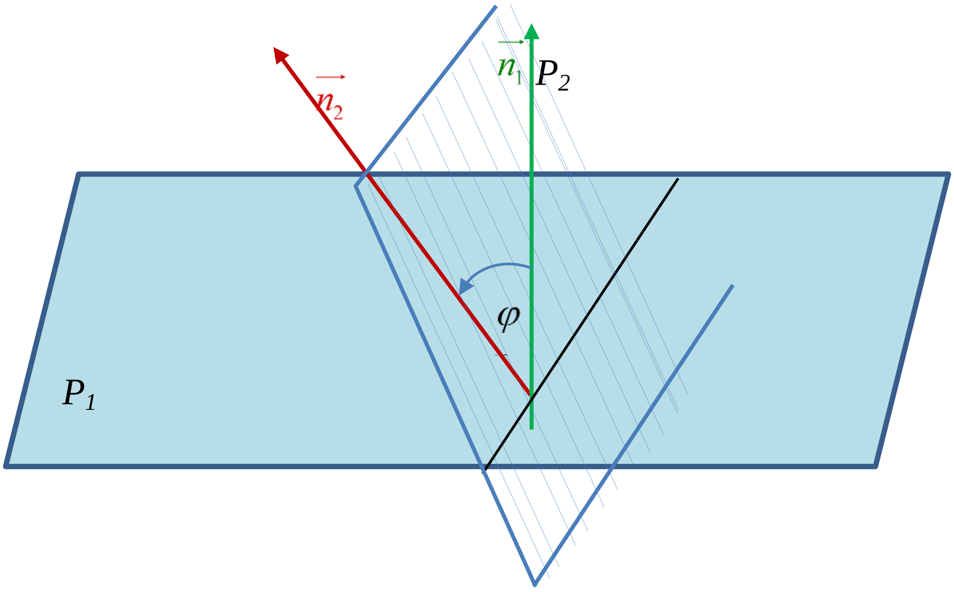
$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} \qquad (2.7)$$

formula bilan aniqlanadi.

Agar

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2} \qquad (2.8)$$

tenglik o‘rinli bo‘lsa, tekisliklar ustma-ust tushadi.



1-chizma.

Tekislikning umumiy tenglamasidagi ba’zi koeffitsientlar nolga aylanganda tekislikning koordinata o‘qlariga nisbatan vaziyati quyidagicha bo‘ladi:

1. Agar $D=0$ bo‘lsa, koordinatalar boshidan o‘tadi.

2. Agar a) $A=0$ bo‘lsa, $\vec{n}=B\vec{j}+C\vec{k}$ normal vektori Ox o‘qiga perpendikulyar bo‘ladi. Demak , tekislik Ox o‘qiga parallel bo‘ladi.

Xuddi shu kabi

b) $B=0$ bo‘lsa, tekislik Oy o‘qiga parallel bo‘ladi;

c) $C=0$ bo‘lsa, tekislik Oz o‘qiga parallel bo‘ladi.

3. Agar a) $D=0$, $C=0$ bo‘lsa, $Ax+By=0$ koordinatalar boshidan o‘tib Oz o‘qiga parallel bo‘ladi. Demak , tekislik Oz o‘qidan o‘tuvchi tekislik bo‘ladi.

Xuddi shu kabi

b) $D=0$, $B=0$ bo‘lsa, tekislik Oy o‘qidan o‘tuvchi tekislik bo‘ladi;

c) $D=0$, $A=0$ bo‘lsa, tekislik Ox o‘qidan o‘tuvchi tekislik bo‘ladi.

4. Agar a) $A=0$, $B=0$ bo‘lsa, $\vec{n}=C\vec{k}$ normal vektori Oz o‘qiga parallel bo‘ladi. Demak , $Cz+D=0$ tekislik Oxy tekisligiga parallel bo‘ladi.

Xuddi shu kabi

b) $A=0$, $C=0$ bo‘lsa, tekislik Oxz tekisligiga parallel bo‘ladi;

c) $B=0$, $C=0$ bo‘lsa, tekislik Oyz tekisligiga parallel bo‘ladi.

5. Agar a) $A=0$, $B=0$ va $D=0$ bo‘lsa, $Cz=0$ yoki $z=0$ tekislik Oxy tekisligiga parallel va koordinata boshidan o‘tadi. Demak, Oxy koordinata tekisligining o‘zi hosil bo‘ladi. Xuddi shu kabi

b) $A=0$, $C=0$ va $D=0$ bo‘lsa, Oxz tekisligi hosil bo‘ladi;

c) $B=0$, $C=0$ va $D=0$ bo‘lsa, Oyz tekisligi hosil bo‘ladi.

Berilgan $M_0(x_0, y_0, z_0)$ nuqtadan $Ax+By+Cz+D=0$ tekislikkacha bo‘lgan d masofa

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}} \tag{2.9}$$

formula bilan hisoblanadi.

1-misol

Agar $M_1(2,0,4)$ va $M_2(5,5,1)$ nuqtalar berilgan bo‘lsa M_1 nuqtadan o‘tuvchi va $\overrightarrow{M_1M_2}$ vektorga perpendikulyar tekislik tenglamasini tuzing.

► $M_1(2,0,4)$ nuqtadan o‘tib, $\overrightarrow{M_1M_2} = \vec{n}(3,5,-3)$ normal vektorga ega bo‘lgan tekislik tenglamasi (2.2) ga ko‘ra,

$$3(x-2)+5(y-0)-3(z-4)=0,$$

$$3x+5y-3z+6=0. \blacktriangleleft$$

2-misol

Ox o‘qiga parallel, hamda $M_1(0,2,11)$ va $M_2(2,3,4)$ nuqtalardan o‘tuvchi tekislik tenglamasini tuzing.

► Ox o‘qiga parallel bo‘lgani uchun tekislikning umumiy tenglamasida $A=0$ bo‘lib, normal vektori $\vec{n}(0;B;C)$ ko‘rinishda bo‘ladi. $\overrightarrow{M_1M_2}(2;1;-7) \perp \vec{n}$ dan va (2.2) formuladan foydalanib quyidagi tenglamalarni tuzamiz:

$$2\cdot 0+1\cdot B-7\cdot C=0,$$

$$B(y-3)+C(z-4)=0.$$

Bu tenglamalarni birgalikda yechib, izlanayotgan tekislik tenglamasini hosil qilamiz.

$7(y-3)+(z-4)=0$ yoki $7y+z-25=0$.◄

3-misol

Berilgan $6x+2y-4z-7=0$ va $9x+3y-6z+13=0$ tekisliklar orasidagi burchakni toping.

► $\overrightarrow{n_1}=\{6,2,-4\}$, $\overrightarrow{n_2}=\{9,3,-6\}$

$$\cos\varphi = \frac{6\cdot 9+2\cdot 3+(-4)\cdot(-6)}{\sqrt{6^2+2^2+(-4)^2}\cdot\sqrt{9^2+3^2+(-6)^2}} = \frac{84}{\sqrt{56}\cdot\sqrt{126}} = \frac{84}{\sqrt{7056}} = \frac{84}{84} = 1,$$

$\varphi = \arccos 1 = 0.$

Demak, berilgan tekisliklar o‘zaro parallel.◄

4-misol.

Berilgan $M_0(4;3;0)$ nuqtadan, berilgan $M_1(1;3;0)$, $M_2(3;0;1)$ va $M_3(4;-1;2)$ nuqtalardan o‘tuvchi tekislikkacha bo‘lgan masofani toping.

► Dastlab (2.4) formuladan foydalanib, uch nuqtadan o‘tuvchi tekislik tenglamasini tuzamiz:

$$\begin{vmatrix} x-1 & y-3 & z-0 \\ 3-1 & 0-3 & 1-0 \\ 4-1 & -1-3 & 2-0 \end{vmatrix} = 0 \quad \text{yoki} \quad \begin{vmatrix} x-1 & y-3 & z \\ 2 & -3 & 1 \\ 3 & -4 & 2 \end{vmatrix} = 0.$$

Determinantni hisoblab, $2x+y-z-5=0$ tekislik tenglamasi hosil qilalamiz. $M_0(4;3;0)$ nuqtadan $2x+y-z-5=0$ tekislikkacha masofa

$$d = \frac{|2\cdot 4+1\cdot 3-1\cdot 0-5|}{\sqrt{2^2+1^2+(-1)^2}} = \frac{6}{\sqrt{6}} = \sqrt{6}. \blacktriangleleft$$

Auditoriya topshiriqlar

1. Quyidagi shartlarni qanoatlantiruvchi

a) berilgan $M_0(2;-3;0)$ nuqtadan o‘tib, $\vec{n}(1,5,-2)$ vektorga perpendikulyar;

b) Berilgan $M_0(3;-1;2)$ nuqtadan o‘tib, Oxz tekisligiga parallel;

c) Berilgan $M_1(1;2;-5)$ va $M_2(2;0;-1)$ va nuqtalardan o‘tib, Oy o‘qiga parallel;

d) $M_0(0;3;4)$ nuqtadan va Oz o‘qidan o‘tuvchi;

e) $A(3;5;-2)$ nuqtadan o‘tib, $\vec{n}_1(2;1;-3)$ va $\vec{n}_2(4;-3;-1)$ vektorlarga parallel tekislik tenglamalarini tuzing va ularni yasang.

2. $M_1(1;2;-5)$ va $M_2(2;0;-1)$ nuqtalardan o‘tib, $3x+5y-3z+6=0$ tekisligiga perpendiulyar tekislik tenglamasini tuzing.

3. $A(4;-3;5)$ nuqtadan o‘tib, koordinata o‘qlaridan $1:2:2$ nisbatdagi musbat kesmalar ajratadigan tekislik tenglamasini tuzing.

4. $7x-y-5z+6=0$ va $2x-y+3z-13=0$ tekisliklar orasidagi burchakni toping.

5. $A(1;3;-5)$ nuqtadan o‘tuvchi va $3x+2y-6z+7=0$, $2x-6y+3z-13=0$ tekisliklarga perpendikulyar tekislik tenglamasini tuzing.

6. $2x+6y-3z+15=0$ va $2x+6y-3z-13=0$ tekisliklar orasidagi masofani toping.

7. $2x-y+4z+21=0$ tekisligiga perpendikulyar va Ox , Oy koordinata o‘qlaridan mos ravishda $a=2$, $b=-3$ kesma ajratuvchi tekislik tenglamasini tuzing.

8. Uchlari $A(-3;0;2)$, $B(1;2;-2)$, $C(0;1;-2)$ va $D(3;-3;2)$ nuqtalarda bo‘lgan piramidaning A uchidan BCD yog‘iga tushirilgan balandligi uzunligini toping.

Mustaqil yechish uchun testlar

1. $A(1;2;1)$ va $B(4;0;-5)$ nuqtalar berilgan. $A(1;2;1)$ nuqtadan o‘tib, $\overrightarrow{AB}$ vektorga perpendikulyar bo‘lgan tekislik tenglamasini toping

$A)$ $2x+3y-6z+2=0$ $B)$ $4x-6y-12z-3=0$

$C)$ $3x-2y-6z-1=0$ $D)$ $6x-2y-3z+5=0$

2. Ox o‘qidan o‘tuvchi tekislik tenglamasi berilgan javobni aniqlang

$A)$ $3y-6z+5=0$ $B)$ $5y+12z=0$

$C)$ $3x-7=0$ $D)$ $6x-2y-3z=0$

3. Oyz koordinata tekisligiga parallel tekislik tenglamasi berilgan javobni aniqlang

$A)$ $3y-6z+5=0$ $B)$ $5y+12z=0$

$C)$ $3x-7=0$ $D)$ $6x-2y-3z=0$

4. $2x-3y+6z-7=0$ tekislikka perpendiklyar tekislikni toping

$A)$ $2x+3y-6z+5=0$ $B)$ $4x-5y+12z-7=0$

$C)$ $3x-2y-6z-7=0$ $D)$ $6x-2y-3z+5=0$

5. Berilgan $M(-9;-1;3)$ nuqtadan $3x+6y+2z-8=0$ tekislikkacha bo‘lgan masofani toping

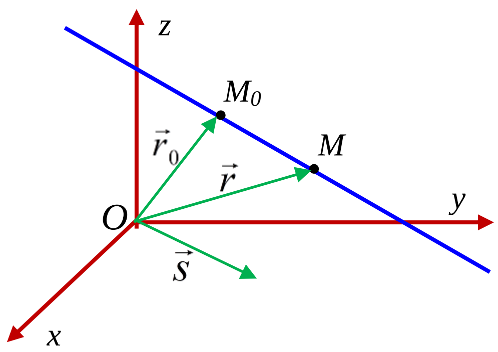
$A)$ 3 $B)$ 4 $C)$ 5 $D)$ 6

6. Berilgan $3x-2y-6z-7=0$ va $6x-3y+2z=0$ tekisliklar orasidagi burchak kosinusini toping

$A)$ $15/49$ $B)$ $18/49$ $C)$ $12/49$ $D)$ $16/49$

3.3 Fazoda to‘g‘ri chiziq. To‘g‘ri chiziq va tekislikning o‘zaro joylashuvi

Agar to‘g‘ri chiziqda yotuvchi $M_0(\vec{r}_0)=M_0(x_0,y_0,z_0)$ nuqta va to‘g‘ri chiziqga parallel $\vec{s}(m,n,p)$, $\left(|\vec{s}|\neq 0\right)$ vektor berilgan bo‘lsa, fazoda to‘g‘ri chiziqning vaziyati aniqlangan bo‘ladi. $M(\vec{r})=M(x,y,z)$ nuqta to‘g‘ri chiziqdagi o‘zgaruvchan nuqta bo‘lsin. U holda $\overrightarrow{M_0M}=t\cdot\vec{s}$ bo‘ladi. Bu yerda t M nuqtaning vaziyatiga qarab ixtiyoriy haqiqiy son qiymati qabul qilishi mumkin. t to‘g‘ri chiziqning o‘zgaruvchan *parametri* deyiladi. $\overrightarrow{M_0M}=\vec{r}-\vec{r}_0$ dan to‘g‘ri chiziqning *vektor tenglamasi* hosil bo‘ladi:



1-chizma

$$\vec{r}=\vec{r}_0+t\cdot\vec{s} \qquad (1)$$

Bu tenglamadan koordinatalarga o‘tsak,

$$\begin{cases} x = x_0 + mt \\ y = y_0 + nt \\ z = z_0 + pt \end{cases} \quad (2)$$

to'g'ri chiziqning parametrik tenglamasi hosil bo'ladi. (2) dan *to'g'ri chiziqning kanonik tenglamasi*ni hosil qilamiz

$$\frac{x - x_0}{m} = \frac{y - y_0}{n} = \frac{z - z_0}{p}. \quad (3)$$

$\vec{s}(m, n, p)$ vektor to'g'ri chiziqning *yo'naltiruvchi vektori* deyiladi.

Ikki $M_1(x_1, y_1, z_1)$ *va* $M_2(x_2, y_2, z_2)$ *nuqtalardan o'tuvchi to'g'ri chiziq* tenglamasi

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \quad (4)$$

Har qanday ikkita parallel bo'lmagan tekisliklarning tenglamalari birgalikda

$$\begin{cases} A_1 x + B_1 y + C_1 z + D_1 = 0 \\ A_2 x + B_2 y + C_2 z + D_2 = 0 \end{cases} \quad (5)$$

to'g'ri chiziqning umumiy tenglamasi deyiladi. To'g'ri chiziqning $\bar{s}$ yo'naltiruvchi vektori sistemadagi tekisliklarning normal vektori $\vec{n_1}(A_1, B_1, C_1)$ va $\vec{n_2}(A_2, B_2, C_2)$ ning har biriga perpendikulyar, demak, $\vec{s} = \vec{n_1} \times \vec{n_2}$.

To'g'ri chiziqning umumiy tenglamasidan kanonik tenglamani hosil qilish mumkin. Buning uchun to'g'ri chiziqda yotuvchi bitta nuqta koordinatalarini va yo'naltiruvchi vektorni bilish yetarli, yoki avval to'g'ri chiziqning proyeksiyalardagi tenglamasiga o'tish lozim.

To'g'ri chiziqning proyeksiyalardagi tenglamasi uning umumiy tenglamasidan avval y ni, keyin x ni yo'qotib topiladi:

$$\begin{cases} x = mz + a \\ y = nz + b. \end{cases} \quad (6)$$

1-misol.

Ushbu $\begin{cases} x - 2y - z - 5 = 0 \\ 2x + y - 3z - 5 = 0 \end{cases}$ umumiy tenglama bilan berilgan to'g'ri chiziqning kanonik tenglamasini yozing.

► Bu yerda $\bar{n}_1(1,-2,-1)$ va $\bar{n}_2(2,1,-3)$, u holda

$$\bar{s} = \bar{n}_1 \times \bar{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & -1 \\ 2 & 1 & -3 \end{vmatrix} = 7\vec{i} + \vec{j} + 5\vec{k}, \qquad \vec{s}(7,1,5).$$

To'g'ri chiziqda yotuvchi bitta nuqtani topish uchun $z = 0$ deb, $x = 3$, $y = -1$ larni topamiz. $M_0(3,-1,0)$ berilgan to'g'ri chiziqda yotadi. Demak, to'g'ri chiziqning kanonik tenglamasi

$$\frac{x-3}{7} = \frac{y+1}{1} = \frac{z}{5}. \quad ◄$$

Ikkita to‘g‘ri chiziq kanonik tenglamalari bilan berilgan bo‘lsin:

$$\frac{x-x_1}{m_1}=\frac{y-y_1}{n_1}=\frac{z-z_1}{p_1}; \qquad \frac{x-x_2}{m_2}=\frac{y-y_2}{n_2}=\frac{z-z_2}{p_2}. \tag{7}$$

Bu *to‘g‘ri chiziqlar orasidagi burchak* ularning yo‘naltiruvchi $\bar{s}_1(m_1,n_1,p_1)$ va $\bar{s}_2(m_2,n_2,p_2)$ vektorlari orasidagi φ burchakka teng

$$\cos\varphi=\pm\frac{\bar{s}_1\cdot\bar{s}_2}{|\bar{s}_1|\cdot|\bar{s}_2|}=\frac{|m_1m_2+n_1n_2+p_1p_2|}{\sqrt{m_1^2+n_1^2+p_1^2}\cdot\sqrt{m_2^2+n_2^2+p_2^2}}. \tag{8}$$

a) to‘g‘ri chiziqlarlarning *perpendikulyarlik sharti*

$$m_1m_2+n_1n_2+p_1p_2=0 \tag{9}$$

b) to‘g‘ri chiziqlarning *parallelik sharti*

$$\frac{m_1}{m_2}=\frac{n_1}{n_2}=\frac{p_1}{p_2} \tag{10}$$

d) to‘g‘ri chiziqlarning *ayqash bo‘lish sharti*

$$\begin{vmatrix} m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \end{vmatrix}\neq 0, \tag{11}$$

e) parallel bo‘lmagan to‘g‘ri chiziqlarning *kesishish sharti*

$$\begin{vmatrix} m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \end{vmatrix}=0. \tag{12}$$

Berilgan $M_1(x_1,y_1,z_1)$ nuqtadan $\bar{s}(m,n,p)$ vektor bo‘ylab yo‘nalgan $M_0(x_0,y_0,z_0)$ nuqtadan o‘tuvchi to‘g‘ri chiziqgacha bo‘lgan masofa

$$d=\left|\vec{s}\times\overrightarrow{M_0M_1}\right|/|\vec{s}| \tag{13}$$

formula bilan hisoblanadi.

2-misol.

Agar $A(0,-2,8)$, $B(4,3,2)$, $C(1,4,3)$ nuqtalar berilgan bo‘lsa, A nuqtadan o‘tib BC to‘g‘ri chiziqga parallel bo‘lgan to‘g‘ri chiziq tenglamasini tuzing.

► Izlanayotgan to‘g‘ri chiziq BC to‘g‘ri chiziqga parallel bo‘lgani uchun $\vec{s}=\overrightarrow{BC}(-3,1,1)$ deb tanlash kifoya. U holda $A(0,-2,8)$ nuqtadan o‘tuvchi yo‘naltiruvchisi $\vec{s}(-3,1,1)$ bo‘lgan to‘g‘ri chiziqning kanonik tenglamasini tuzamiz:

$$\frac{x}{-3}=\frac{y+2}{1}=\frac{z-8}{1} \blacktriangleleft$$

3-misol.

Berilgan $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z}{2}$ va $\frac{x-7}{3}=\frac{y-1}{4}=\frac{z-3}{2}$ to‘g‘ri chiziqlar orasidagi masofani toping.

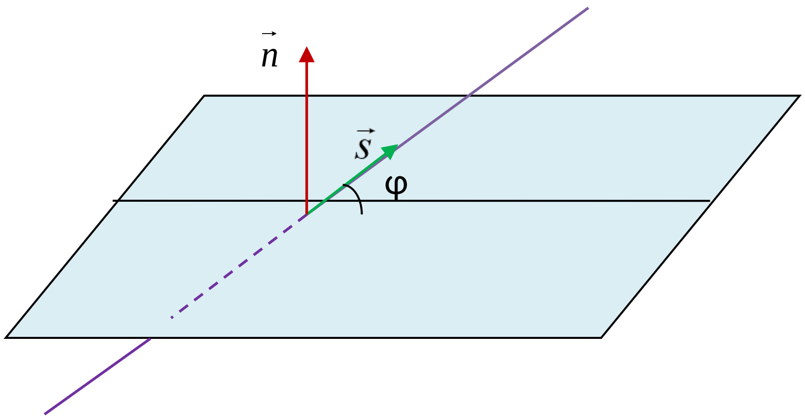
►Birinchi to‘g‘ri chiziqda yotgan ixtiyoriy nuqtadan, masalan, $M_1(2,-1,\ 0)$ dan ikkinchi $\frac{x-7}{3}=\frac{y-1}{4}=\frac{z-3}{2}$ to‘g‘ri chiziqgacha masofa topiladi.

$$\overrightarrow{M_1M_0}(5,\ 2,\ 3),\ \vec{s}(3,\ 4,\ 2),\ |\vec{s}|=\sqrt{3^2+4^2+2^2}=\sqrt{29}\ ,$$

$$\vec{s}\times\overrightarrow{M_1M_0}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & 4 & 2\\ 5 & 2 & 3\end{vmatrix}=8\vec{i}+\vec{j}-14\vec{k},\ \left|\vec{s}\times\overrightarrow{M_1M_0}\right|=3\sqrt{29}\,.$$

To‘g‘ri chiziqlar orasidagi masofa $d=\left|\vec{s}\times\overrightarrow{M_1M_0}\right|/|\vec{s}|=3$. ◄

To‘g‘ri chiziq (L): $\frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p}$ va tekislik $(T):\ Ax+By+Cz+D=0$ tenglamalari berilgan bo‘lsin. To‘g‘ri chiziq va tekislik orasidagi *burchak* deb, to‘g‘ri chiziq va uning tekislikdagi orthogonal proyeksiyasi orasidagi φ burchakga aytiladi.



To‘g‘ri chiziq va tekislik orasidagi burchak quyidagi formula bilan hisoblanadi:

$$\left|\cos\left(\widehat{\vec{n},\vec{s}}\right)\right|=\sin\varphi=\frac{|Am+Bn+Cp|}{\sqrt{A^2+B^2+C^2}\sqrt{m^2+n^2+p^2}}.$$

To‘g‘ri chiziqning kanonik tenglamasidan parametrik tenglamasiga o‘tib, tekislik tenglamasiga qo‘yamiz

$$(Am+Bn+Cp)t+Ax_0+By_0+Cz_0+D=0.$$

Bunda uch hol bo‘lishi mumkin.

1. Agar $Am+Bn+Cp\neq 0$ bo‘lsa, *to‘g‘ri chiziq va tekislik kesishadi.* Bu holda $t=-(Ax_0+By_0+Cz_0+D)/(Am+Bn+Cp)$ ni to‘g‘ri chiziq parametrik tenglamasiga qo‘yib, *to‘g‘ri chiziq va tekislikning kesishish nuqtasi* M topiladi.

Xususan, $\frac{A}{m}=\frac{B}{n}=\frac{C}{p}$ bo‘lsa, *to‘g‘ri chiziq va tekislik perpendikulyar* bo‘ladi.

2. Agar $Am+Bn+Cp=0$ va $Ax_0+By_0+Cz_0+D\neq 0$ bo‘lsa, *to‘g‘ri chiziq va tekislik parallel*.

3. Agar $Am+Bn+Cp=0$ va $Ax_0+By_0+Cz_0+D=0$ bo‘lsa, *to‘g‘ri chiziq tekislikda yotadi*(to‘g‘ri chiziq tekislikga tegishli).

4-misol.

Berilgan $\frac{x-1}{-1}=\frac{y-1,5}{0}=\frac{z-3}{1}$ to‘g‘ri chiziqqa nisbatan $M(3,3,3)$ nuqtaga simmetrik $M`$ nuqtani toping.

► $M(3,3,3)$ nuqtadan o‘tuvchi $\frac{x-1}{-1}=\frac{y-1,5}{0}=\frac{z-3}{1}$ to‘g‘ri chiziqga perpendikulyar tekislik tenglamasini topamiz.

$$-1(x-3)+0(y-3)+1(z-3)=0,$$

$$-x+z=0.$$

To‘g‘ri chiziq va tekislik kesishgan nuqtani topamiz.

$$\frac{x-1}{-1}=\frac{y-1,5}{0}=\frac{z-3}{1}\Rightarrow\begin{cases}x=-t+1,\\ y=1,5,\\ z=t+3.\end{cases}$$

$$-(-t+1)+(t+3)=0,$$
$$2t+2=0,$$
$$t=-1.$$

$M_0(2;1,5;2)$ - kesishish nuqtasi. Bundan

$$x_{M_0}=\frac{x_M+x_{M'}}{2}\Rightarrow x_{M'}=2x_{M_0}-x_M=2\cdot2-3=1,$$
$$y_{M_0}=\frac{y_M+y_{M'}}{2}\Rightarrow y_{M'}=2y_{M_0}-y_M=2\cdot1,5-3=0,$$
$$z_{M_0}=\frac{z_M+z_{M'}}{2}\Rightarrow z_{M'}=2z_{M_0}-z_M=2\cdot2-3=1.$$

Natijada, $M'(1,0,1)$ izlangan nuqtaga ega bo‘lamiz. ◄

Auditoriya topshiriqlari

1. $\begin{cases}x-3y+2z+2=0\\ x+3y+z+14=0\end{cases}$ umumiy tenglama bilan berilgan to‘g‘ri chiziqning kanonik tenglamasini yozing. (Javob: $\frac{x+8}{-9}=\frac{y+2}{1}=\frac{z}{6}$.)

2. Uchburchakning $A(1,-2,3)$, $B(4,3,-2)$, $C(2,1,-2)$ uchlari berilgan bo‘lsa, AD medianasining parametrik tenglamasini yozing.(Javob: $x=1+3t, y=-2+2t, z=3-2t$.)

3. A va B ning qanday qiymatlarida $Ax+By+6z-5=0$ tekislik va $\frac{x-3}{2}=\frac{y+4}{-5}=\frac{z+2}{3}$ to‘g‘ri chiziq perpendikulyar bo‘ladi? (Javob: $A=4,\ B=-10.$)

4. To‘g‘ri chiziq va tekislik orasidagi burchakni toping:

$a)\ \begin{cases} 3x - y - 1 = 0, \\ 3x + 2z - 2 = 0 \end{cases}$ va $2x + y + z - 4 = 0;$

$b)\ \begin{cases} x - 2y + 3 = 0, \\ 3y - z - 1 = 0 \end{cases}$ va $2x + 3y + z + 1 = 0.$

(Javob: $a)\ \varphi = \arcsin \frac{1}{\sqrt{6}}$; $b)\ \varphi = \arcsin \frac{5}{7}$.)

5. To‘g‘ri chiziq va tekislikning o‘zaro joylashuvini aniqlang. Agar ular kesishuvchi bo‘lsa, kesishish nuqtasini toping:

$a)\ \frac{x-3}{2} = \frac{y+4}{4} = \frac{z}{3}$ va $3x - 3y + 2z - 5 = 0;$

$b)\ \frac{x-13}{5} = \frac{y-1}{2} = \frac{z-4}{3}$ va $x + 2y - 3z - 3 = 0;$

$d)\ \frac{x-5}{1} = \frac{y-4}{1} = \frac{z-7}{3}$ va $2x - y + 3z - 7 = 0.$

(Javob: $a)$ parallel; $b)$ to‘g‘ri chiziq tekislikda yotadi; $d)$ $M(3{,}2,\ 1)$ nuqtada kesishadi.)

6. $A(3,\ 4,\ 0)$ nuqtadan va $\frac{x-2}{1} = \frac{y-3}{2} = \frac{z+1}{3}$ to‘g‘ri chiziqdan o‘tuvchi tekislik tenglamasini yozing. (Javob: $x - 2y + z + 5 = 0$.)

7. $\frac{x-2}{1} = \frac{y+3}{2} = \frac{z+1}{3}$ to‘g‘ri chiziqdan o‘tuvchi va $2x - y - z - 3 = 0$ tekislikga perpendikulyar tekislik tenglamasini yozing. (Javob: $x + 7y - 5z + 14 = 0$.)

8. $\frac{x+3}{2} = \frac{y-2}{-3} = \frac{z+1}{1}$ va $\frac{x-2}{2} = \frac{y+3}{-3} = \frac{z}{1}$ parallel to‘g‘ri chiziqlardan o‘tuvchi tekislik tenglamasini yozing. (Javob: $2x + 3y + 5z + 5 = 0$.)

9. $A(3,\ 1, -1)$ nuqtaning $x - 2y + z + 6 = 0$ tekislikdagi proyeksiyasini toping. (Javob: (2, 3, -2).)

10. $A(3,\ 1, -2)$ nuqtaning $\frac{x+3}{2} = \frac{y-2}{-3} = \frac{z+1}{1}$ to‘g‘ri chiziqdagi proyeksiyasini toping. (Javob: (-1, -1, 0).)

11. $\begin{cases} x = z - 2 \\ y = 2z + 1 \end{cases}$ va $\frac{x-2}{3} = \frac{y-4}{1} = \frac{z-2}{1}$ to‘g‘ri chiziqlarning kesishuvchi ekanligini ko‘rsating, hamda ular joylashgan tekislik tenglamasini yozing. (Javob: $x + 2y - 5z = 0$.)

Mustaqil yechish uchun testlar

1. $A(3,\,-2,\,0)$ va $B(5,\,-4,\,3)$ nuqtalardan oʻtuvchi toʻgʻri chiziqning parametrik tenglamasini yozing.

A) $\begin{cases} x=3+3t, \\ y=-2+2t, \\ z=-2t. \end{cases}$ B) $\begin{cases} x=5+2t, \\ y=-4-2t, \\ z=3+3t. \end{cases}$ C) $\begin{cases} x=3+2t, \\ y=-2+2t, \\ z=3t. \end{cases}$ D) $\begin{cases} x=1+3t, \\ y=-2+2t, \\ z=3-2t. \end{cases}$

2. A ning qanday qiymatida $\frac{x-2}{3}=\frac{y+1}{A}=\frac{z}{2}$ va $\frac{x-7}{3}=\frac{y-1}{5}=\frac{z-3}{-2}$ toʻgʻri chiziqlar perpendikulyar boʻladi?

A) 1; B) -2; C) 3; D) -1.

3. $\frac{x-2}{1}=\frac{y+3}{2}=\frac{z+1}{3}$ toʻgʻri chiziq va $x+7y-5z+14=0$ tekislik qanday joylashgan?

A) parallel; B) perpendikulyar; C) toʻgʻri chiziq tekislikda yotadi; D) kesishadi.

4. $\frac{x+1}{3}=\frac{y+1}{-2}=\frac{z+2}{2}$ toʻgʻri chiziq va $2x+3y+5z+5=0$ tekislik kesishgan nuqtani toping.

A) $(3,\,-2,\,0)$, B) $(3,\,-2,\,-1)$, C) $(4,\,-1,\,-2)$, D) $(2,\,-3,\,0)$.

5. $\frac{x+2}{-1}=\frac{y-3}{2}=\frac{z-4}{3}$ va $\frac{x-3}{3}=\frac{y+2}{2}=\frac{z-8}{5}$ toʻgʻri chiziqlar qanday joylashgan?

A) parallel; B) perpendikulyar; C) ayqash; D) kesishadi.

Shaxsiy uy topshiriqlari

1.1. $M(4,\,-1,\,-2)$ nuqtadan oʻtuvchi va $2x-y-3z+5=0$ tekislikga parallel boʻlgan tekislikning oʻqlardan ajratgan kesmalarini toping.

1.2. $A(-1,\,3,\,2)$, $B(1,1,0)$ nuqtalardan oʻtuvchi va $x+2y-3z-3=0$ tekislikga perpendikulyar boʻlgan tekislik tenglamasini yozing.

1.3. Agar $M_1(3,-2,4)$, $M_2(-1,\,4,\,2)$ nuqtalar berilgan boʻlsa, M_1M_2 kesmaning oʻrtasidan oʻtuvchi va shu kesmaga perpendikulyar tekislik tenglamasini yozing.

1.4. Ox oʻqidan va $A(-1,\,3,\,-3)$ nuqtadan oʻtuvchi tekislik tenglamasini yozing va $x-2y+2z+5=0$ tekislik bilan hosil qilgan burchagini aniqlang.

1.5. $M(4,\,-1,\,-2)$ nuqtadan $2x+2y-z+4=0$ tekislikgacha boʻlgan masofani toping.

1.6. $A(-1,\,3,\,2)$, $B(1,1,0)$ va $C(2,\,0,\,-1)$ nuqtalardan oʻtuvchi tekislik tenglamasini yozing.

1.7. $A(4,\,1,\,2)$, $B(2,-1,3)$ nuqtalardan oʻtuvchi va $\vec{a}(1,2,-5)$ vektorga parallel boʻlgan tekislik tenglamasini yozing.

1.8. $A(3,\,2,-3)$, $B(-1,\,4,\,2)$ nuqtalardan oʻtuvchi va Oy oʻqiga parallel boʻlgan tekislik tenglamasini yozing.

1.9. $M(5,\ 4,\ -8)$ nuqtadan $3x+6y-2z+15=0$ tekislikgacha bo‘lgan masofani toping.

1.10. $A(1,\ 2,\ 1),\ B(3,0,3)$ nuqtalardan o‘tuvchi va Ox o‘qidan $a=2$ kesma ajratuvchi tekislik tenglamasini yozing.

1.11. $A(-1,\ 2,\ 3)$ nuqtadan o‘tuvchi, $3x-y+2z+7=0$ va $2x+y+3z-5=0$ tekisliklarga perpendikulyar bo‘lgan tekislik tenglamasini yozing.

1.12. $A(2,\ -5,\ 2),\ B(1,0,1)$ va $C(2,\ 4,\ -1)$ nuqtalardan o‘tuvchi tekislik tenglamasini yozing.

1.13. O‘zaro parallel bo‘lgan $2x-9y+6z+17=0$ va $2x-9y+6z-16=0$ tekisliklar orasidagi masofani toping.

1.14. $x-3y+6=0$ va $x+2y-7=0$ tekisliklar orasidagi burchakni toping.

1.15. Oz o‘qidan o‘tuvchi va $2x+y-2z+7=0$ tekislik bilan 45° burchak tashkil etuvchi tekislik tenglamasini yozing.

1.16. $3x+6y-2z+15=0$ tekislikdan 4 birlik masofada yotuvchi tekislik tenglamasini yozing.

1.17. $C(2,\ 0,\ -1)$ nuqtadan o‘tuvchi va $\vec{a}(1,\ 3,-2),\ \vec{b}(1,-1,1)$ vektorlarga perpendikulyar tekislik tenglamasini yozing.

1.18. $x-2y-z-14=0$ va $x+y+z-3=0$ tekisliklarning kesishish chizig‘idan hamda $A(2,\ 4,\ -2)$ nuqtadan o‘tuvchi tekislik tenglamasini yozing.

1.19. $x-2y+z-7=0$, $2x+y-3z+16=0$ tekisliklarning kesishish chizig‘idan o‘tuvchi va $4x+3y+z-15=0$ tekislikga perpendikulyar tekislik tenglamasini yozing.

1.20. $A(-1,\ 3,\ 2),\ B(1,1,0)$ va $C(2,\ 0,\ -1)$ nuqtalardan o‘tuvchi tekislik bilan Oxz tekislik orasidagi burchakni toping.

1.21. $2x-y+2z-7=0$ va $x-2y+2z-2=0$ tekisliklarning kesishish chizig‘idan o‘tuvchi hamda Ox o‘qiga parallel bo‘lgan tekislik tenglamasini yozing.

1.22. O‘zaro parallel bo‘lgan $2x-3y+6z-3=0$ va $2x-3y+6z-24=0$ tekisliklar orasidagi masofani toping.

1.23. $A(2,\ 1,\ 3)$ nuqtadan o‘qlardan $a=1,\ b=2, c=3$ kesma ajratuvchi tekislikgacha bo‘lgan masofani toping.

1.24. Ox o‘qidan o‘tuvchi va $2x+y-2z+7=0$ tekislik bilan 45° burchak tashkil etuvchi tekislik tenglamasini yozing.

1.25. $A(2,\ -3,\ 1)$ nuqtadan o‘tuvchi, $2x+y-2z+7=0$ va $2x+y+3z-5=0$ tekisliklarga perpendikulyar bo‘lgan tekislik tenglamasini yozing.

1.26. $M(3,\ -1,\ 7)$ nuqtadan o‘tuvchi va $3x+y-2z+15=0$ tekislikga parallel bo‘lgan tekislikning o‘qlardan ajratgan kesmalarini toping.

1.27. $A(2,\ -3,-1)$ nuqtadan o‘tuvchi, $x-3y+6=0$ va $2x+y-2z-5=0$ tekisliklarga perpendikulyar bo‘lgan tekislik tenglamasini yozing.

1.28. $M(-3,\ 1,\ -9)$ nuqtaning $4x-3y-z-7=0$ tekislikga nisbatan simmetrik bo‘lgan M' nuqta koordinatalarini toping.

1.29. $5x+3y+z-18=0$ va $2x+z-9=0$ tekisliklar orasidagi burchakni toping.

1.30. O‘zaro parallel bo‘lgan $3x-2y-6z-13=0$ va $3x-2y-6z+15=0$ tekisliklar orasidagi masofani toping.

2.Quyidagi umumiy tenglama bilan berilgan to‘g‘ri chiziqlarning kanonik tenglamalarini yozing.

2.1 $\begin{cases}2x+y+z-2=0\\2x-y-3z+6=0\end{cases}$ 2.2 $\begin{cases}6x-7y-4z-2=0\\x+7y-z-5=0\end{cases}$

2.3 $\begin{cases}x+3y+z+14=0\\x-3y+2z+2=0\end{cases}$ 2.4 $\begin{cases}x-2y+3z-2=0\\2x+3y-8z+3=0\end{cases}$

2.5 $\begin{cases}x+y+z-2=0\\x-y-2z+2=0\end{cases}$ 2.6 $\begin{cases}6x-5y-4z+8=0\\6x+5y+3z+4=0.\end{cases}$

2.7 $\begin{cases}2x+2y-z-8=0\\x-2y+z-4=0\end{cases}$ 2.8 $\begin{cases}2x-5y+2z+5=0\\x+5y-z-5=0.\end{cases}$

2.9 $\begin{cases}3x+y-z-6=0\\3x-y+2z=0\end{cases}$ 2.10 $\begin{cases}2x-3y+z+6=0\\x-3y-2z+3=0\end{cases}$

2.11 $\begin{cases}x+5y+2z+11=0\\x-y-z-1=0\end{cases}$ 2.12 $\begin{cases}4x+y+z+2=0\\2x-y-3z-5=0\end{cases}$

2.13 $\begin{cases}2x+3y+z+6=0\\x-3y-2z+3=0\end{cases}$ 2.14 $\begin{cases}2x+y-3z-2=0\\2x-y+z+6=0\end{cases}$

2.15 $\begin{cases}3x+4y-2z+1=0\\2x-4y+3z+4=0.\end{cases}$ 2.16 $\begin{cases}x+y-2z-2=0\\x-y+z+2=0\end{cases}$

2.17 $\begin{cases}5x+y-3z+4=0\\x-y+2z+2=0.\end{cases}$ 2.18 $\begin{cases}x+5y-z++11=0\\x-y+2z-1=0.\end{cases}$

2.19 $\begin{cases}x-y-z-2=0\\x-2y+z+4=0.\end{cases}$ 2.20 $\begin{cases}x-2y-z+4=0\\x-y+z-2=0\end{cases}$

2.21 $\begin{cases}4x+y-3z+2=0\\2x-y+z-8=0.\end{cases}$ 2.22 $\begin{cases}6x-7y-z-2=0\\x+7y-4z-5=0\end{cases}$

2.23 $\begin{cases}3x+3y-2z-1=0\\2x-3y+z+6=0.\end{cases}$ 2.24 $\begin{cases}x+5y+2z-2=0\\2x-5y-z+5=0\end{cases}$

2.25 $\begin{cases}6x-7y-4z-2=0\\x+7y-z-5=0\end{cases}$ 2.26 $\begin{cases}x+3y+2z+14=0\\x-3y+z+2=0.\end{cases}$

2.27 $\begin{cases}2x+3y-2z+6=0\\x-3y+z+3=0.\end{cases}$ 2.28 $\begin{cases}3x+3y+z-1=0\\2x-3y-2z+6=0.\end{cases}$

2.29 $\begin{cases}3x+4y+3z+1=0\\2x-4y-2z+3=0\end{cases}$ 2.30 $\begin{cases}6x+5y-4z+4=0\\6x-5y+3z+8=0.\end{cases}$

3. Quyidagi masalalarni yeching.

3.1. $M(3,\,-1,\,7)$ nuqtadan o‘tuvchi va $\begin{cases} 3x+y+3z+1=0 \\ x-2y-z+4=0 \end{cases}$ to‘g‘ri chiziqga parallel to‘g‘ri chiziq tenglamasini toping.(Javob: $\frac{x-3}{5}=\frac{y+1}{6}=\frac{z-7}{-7}$.)

3.2. m va C ning qanday qiymatlarida $\frac{x-3}{m}=\frac{y+2}{2}=\frac{z-8}{-5}$ to‘g‘ri chiziq $3x-2y+Cz-7=0$ tekislikga perpendikulyar bo‘ladi? .(Javob: $m=-3, C=5$.)

3.3. p ning qanday qiymatida $\frac{x-3}{1}=\frac{y+2}{2}=\frac{z-4}{p}$ va $\begin{cases} 3x+y-5z+1=0 \\ x-2y+3z-2=0 \end{cases}$ to‘g‘ri chiziqlar perpendikulyar bo‘ladi? .(Javob: $p=-5$.)

3.4. $\frac{x-3}{2}=\frac{y+2}{-1}=\frac{z+1}{-1}$ to‘g‘ri chiziqdan o‘tuvchi va $x+2y+3z-5=0$ tekislikga perpendikulyar tekislik tenglamasini yozing. (Javob: $x+7y-5z+6=0$.)

3.5. D ning qanday qiymatida $\begin{cases} x+y-2z+D=0 \\ x-2y+3z+12=0 \end{cases}$ to‘g‘ri chiziq Oz o‘qini kesib o‘tadi? (Javob: $D=-8$.)

3.6. $M(1,\,3,-2)$ nuqtaning $\begin{cases} x+2y-z+3=0 \\ x+y+z+2=0 \end{cases}$ to‘g‘ri chiziqga nisbatan simmetrik nuqtasini toping.(Javob: $(-3,\,-5,\,2)$.)

3.7. $M(2,\,0,\,1)$ nuqtadan va $\frac{x-3}{3}=\frac{y+1}{5}=\frac{z+1}{-2}$ to‘g‘ri chiziqdan o‘tuvchi tekislik tenglamasini yozing. (Javob: $3x-y+2z-8=0$.)

3.8. $\frac{x-3}{3}=\frac{y+1}{5}=\frac{z+1}{-2}$ va $\frac{x+3}{3}=\frac{y+1}{-5}=\frac{z-8}{-7}$ to‘g‘ri chiziqlarning kesishuvchi ekanini isbotlang va shu to‘g‘ri chiziqlardan o‘tuvchi tekislik tenglamasini yozing.(Javob: $3x-y+2z-8=0$.)

3.9. $\frac{x+6}{3}=\frac{y-4}{-5}=\frac{z-15}{-7}$ va $\begin{cases} 2x+3z-3=0 \\ 2y+5z+7=0 \end{cases}$ to‘g‘ri chiziqlarning kesishish nuqtasini toping. (Javob: $(0,-6,\,1)$.)

3.10. $M(3,\,0,-2)$ nuqtadan $\frac{x-1}{2}=\frac{y+6}{-1}=\frac{z-2}{3}$ to‘g‘ri chiziqga tushirilgan perpendikulyar tenglamasini yozing. (Javob: $\frac{x-3}{4}=\frac{y}{5}=\frac{z+2}{-1}$.)

3.11. $\frac{x-3}{3}=\frac{y-2}{2}=\frac{z+5}{-3}$ va $\frac{x-2}{3}=\frac{y-1}{2}=\frac{z+3}{-3}$ parallel to‘g‘ri chiziqlardan o‘tuvchi tekislik tenglamasini yozing.(Javob: $x-3y-z-2=0$.)

3.12. $\frac{x+2}{2}=\frac{y-1}{3}=\frac{z+6}{6}$ va $\begin{cases}4x-y-z-2=0\\2x+y-2z+17=0\end{cases}$ to‘g‘ri chiziqlar orasidagi burchakni toping. (Javob: $\varphi=\arccos\frac{20}{21}\approx 17°48'$.)

3.13. $M(3,\ 4,\ 0)$ nuqtadan va $\frac{x+1}{1}=\frac{y+2}{2}=\frac{z+1}{2}$ to‘g‘ri chiziqgacha bo‘lgan masofani toping. (Javob: $d=\sqrt{17}.$)

3.14. $\frac{x-1}{1}=\frac{y+1}{-2}=\frac{z-3}{3}$ va $\begin{cases}x-2y+3z-2=0\\2x+3y-8z+3=0\end{cases}$ to‘g‘ri chiziqlar orasidagi burchakni toping. (Javob: $\varphi=90°$.)

3.15. $A(-5,\ -3,\ 2)$ nuqtaning $\frac{x+1}{2}=\frac{y+1}{-3}=\frac{z}{1}$ to‘g‘ri chiziqga nisbatan simmetrik nuqtasini toping. (Javob: (3, 1,-2).)

3.16. $M(3,\ -1,\ 5)$ nuqtadan o‘tuvchi va $\begin{cases}3x+y+3z+1=0\\x-2y-z+4=0\end{cases}$ to‘g‘ri chiziqga perpendikulyar tekislik tenglamasini toping. (Javob: $5x+6y-7z+26=0$.)

3.17. $M(5,-1,3)$ nuqtaning $3x-y+2z-8=0$ tekislikdagi proyeksiyasini toping.(Javob: $(2,\ 0,\ 1).$)

3.18. $\frac{x}{3}=\frac{y-1}{3}=\frac{z-1}{-1}$ to‘g‘ri chiziqdan o‘tuvchi va $x-7y+5z-5=0$ tekislikga perpendikulyar tekislik tenglamasini yozing. (Javob: $x-2y-3z+5=0$.)

3.19. $x=2t+1, y=4t+2, z=5t+3$ to‘g‘ri chiziqga nisbatan $M(4,3,10)$ nuqtaga simmetrik bo‘lgan M' nuqtani toping. (Javob: $M'(2,9,6)$.)

3.20. $\frac{x-5}{2}=\frac{y+3}{-1}=\frac{z+2}{-1}$ to‘g‘ri chiziqdan va $M(4,-1,3)$ nuqtadan o‘tuvchi tekislik tenglamasini yozing. (Javob: $x+3y-z+2=0$.)

3.21. $\frac{x-3}{2}=\frac{y+1}{-1}=\frac{z-2}{-1}$ to‘g‘ri chiziqdan va $M(3,1,8)$ nuqtadan o‘tuvchi tekislik tenglamasini yozing. (Javob: $x+3y-z+2=0$.)

3.22. $\frac{x+4}{3}=\frac{y-2}{1}=\frac{z}{1}$ va $\begin{cases}x-y+z+3=0\\3x-y-z+7=0\end{cases}$ to‘g‘ri chiziqlar kesishuvchi ekanini isbotlang, kesishish nuqtasini toping. (Javob: $(-1,3,1)$.)

3.23. $\begin{cases}x+y-z-5=0\\x-2y+z=0\end{cases}$ to‘g‘ri chiziq bilan $x+2y+3z-30=0$ tekislik perpendikulyar ekanini isbotlang va kesishish nuqtasini toping. (Javob: $(5,5,5)$.

3.24. $\begin{cases}2x-y-2z-2=0\\x-y-4=0\end{cases}$ va $\begin{cases}3x-2y-2z+2=0\\y-2z-1=0\end{cases}$ to‘g‘ri chiziqlar o‘zaro parallel ekanini isbotlang, ular orasidagi masofani toping. (Javob: $d=\sqrt{17}.$)

3.25. $M(1,-4,-5)$ nuqtadan $x=4t+6, y=3t+4, z=2t+2$ to‘g‘ri chiziqgacha bo‘lgan masofani toping. (Javob: $\sqrt{22}.$)

3.26. $A(2,\ 6,\ 9)$ nuqtaning $\begin{cases} x-2y+2z+1=0 \\ 3x-2y+z+1=0 \end{cases}$ to‘g‘ri chiziqdagi proyeksiyasini toping. (Javob: $(3,8,6).$)

3.27. $\begin{cases} x+2y+2z-1=0 \\ 3x+y-4z+2=0 \end{cases}$ to‘g‘ri chiziq bilan $A(2,-2,0)$ va $B(3,-3,-1)$ nuqtalardan o‘tuvchi to‘g‘ri chiziq orasidagi burchakni toping. (Javob: $\varphi=\arccos\dfrac{\sqrt{3}}{3}.$)

3.28. $x=2t-3,\ y=3t+1,\ z=-t-2$ to‘g‘ri chiziqdan o‘tuvchi va $3x-2y+z=0$ tekislikga perpendikulyar tekislik tenglamasini yozing. (Javob: $x-5y-13z-18=0$.)

3.29. $A(2,\ 1,-3)$ nuqtadan o‘tub, $\begin{cases} x-2y+2z=0 \\ 3x-2y+z+1=0 \end{cases}$ to‘g‘ri chiziqga parallel bo‘lgan to‘g‘ri chiziqning parametrik tenglamasini yozing. (Javob: $x=2t+2, y=5t+1, z=4t-3.$)

3.30. $\begin{cases} 5x-2y+2=0 \\ 2x-z+1=0 \end{cases}$ to‘g‘ri chiziq va $A(4,\ 6,\ 1)$ va $B(0,\ -4,-7)$ nuqtalardan o‘tuvchi to‘g‘ri chiziqlarning parallelligini isbotlang va ulardan o‘tuvchi tekislik tenglamasini yozing. (Javob: $10x-8y+5z+3=0.$)

IV-BOB MATEMATIK ANALIZ ASOSLARI

4.1 Kompleks sonlar va ular ustida amallar. Muavr va Eyler formulalari

Ushbu $z = x + iy$ ko‘rinishdagi songa *kompleks son* deyiladi, bu yerda x, y - haqiqiy sonlar, i esa $i^2 = -1$ bo‘lgan *mavhum birlik*. x- kompleks sonning *haqiqiy qismi*, y esa *mavhum qismi* deb ataladi va $\operatorname{Re} z = x$, $\operatorname{Im} z = y$ kabi belgilanadi. Agar $y = 0$ bo‘lsa, $z = x \in R$, agar $x = 0$ bo‘lsa, $z = iy$ *sof mavhum son* hosil bo‘ladi.

Geometrik nuqtai nazardan, har bir $z = x + iy$ kompleks songa koordinatalar tekisligida bitta M(x, y) nuqta (yoki $\overrightarrow{OM}$ vektor) va, aksincha, har bir M(x, y) nuqtaga bitta $z = x + iy$ kompleks son mos keladi. Barcha kompleks sonlar to‘plami C harfi bilan belgilanadi va $R \subset C$.

$z = x + iy$ va $\bar{z} = x - iy$ sonlar *qo‘shma kompleks sonlar* deyiladi.

$z_1 = x_1 + iy_1$ va $z_2 = x_2 + iy_2$ ikkita kompyleks sonlar uchun quyidagi amallar o‘rinli:

1) $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$;

2) $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$;

3) $\dfrac{z_1}{z_2} = \dfrac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} = \dfrac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\dfrac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$.

Ma’lumki, har bir $z = x + iy$ kompleks son uchun $x = r\cos\varphi$, $y = r\sin\varphi$ formulalar o‘rinli. $r = |\overrightarrow{OM}| = \sqrt{x^2 + y^2}$ son $z = x + iy$ *kompleks sonning moduli* deyiladi, $\overrightarrow{OM}$ vektor va Ox o‘qining musbat yo‘nalishi bilan hosil qilgan φ burchagi esa *kompleks sonning argumenti* deyiladi va $\varphi = \arg z$ kabi belgilanadi. U quyidagi

$$\varphi = \begin{cases} arctg\,\dfrac{y}{x}, \text{ agar } x > 0,\ y > 0 \text{ bo`lsa;} \\ \pi + arctg\,\dfrac{y}{x}, \text{ agar } x < 0 \text{ bo`lsa;} \\ 2\pi + arctg\,\dfrac{y}{x}, \text{ agar } x > 0,\ y < 0 \text{ bo`lsa} \end{cases} \tag{1.1}$$

formula bilan hisoblanadi. Har qanday $z = x + iy$ kompleks son

$$z = r(\cos\varphi + i\sin\varphi) \tag{1.2}$$

trigonometrik shaklda yoki $z = re^{i\varphi}$ *ko‘rsatkichli shakl*da ifodalanadi(chunki $e^{i\varphi} = \cos\varphi + i\sin\varphi$ Eyler formulasi o‘rinli).

Agar $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$, $z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$ kompleks sonlar bo‘lsa,

$$z_1 z_2 = r_1 r_2(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)); \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)). \tag{1.3}$$

$z = r(\cos\varphi + i\sin\varphi)$ kompleks sonni n-*darajaga oshirish* uchun ***Muavr formulasi***

$$z^n = r^n(\cos n\varphi + i\sin n\varphi) \tag{1.4}$$

o‘rinli. *n-ildiz chiqarish* uchun esa

$$z_k = \sqrt[n]{z} = \sqrt[n]{r}(\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi + 2k\pi}{n}), \quad k = 0,1,2,...,n-1 \qquad (1.5)$$

formula qo‘llanadi.

Auditoriya topshiriqlari

1. $z_1 = 2+3i$, $z_2 = 3-4i$ va $z_3 = 5-2i$ bo‘lsa, $(2z_1 + z_2)z_3$ ni hisoblang. (Javob: $39-4i$.)

2. $z_1 = 2+3i$, $z_2 = 3-4i$ va $z_3 = 1-2i$ bo‘lsa, $((z_1 + z_3)z_2)/z_3$ ni hisoblang. (Javob: $\frac{31}{5} + \frac{17}{5}i$.)

3. Kompleks sonlarni trigonometrik shaklda ifodalang: $a)\ z_1 = -1+i$, $b)\ z_2 = 3i$ $d)\ z_3 = 2-2i$, $e)\ z_4 = -4$.

4. $z_1 = -2+3i$ bo‘lsa, $|z - z_1| < 1$ shartni qanoatlantiruvchi nuqtalarning geometrik o‘rni qanday sohani ifodalaydi? (Javob: Markazi z_1 nuqtada bo‘lgan R=1 radiusli doiraning ichki qismi.)

5. $z_1 = -1+3i$ bo‘lsa, $|z + z_1| > 2$ shartni qanoatlantiruvchi nuqtalarning geometrik o‘rni qanday sohani ifodalaydi? (Javob: Markazi $-z_1$ nuqtada bo‘lgan R=2 radiusli doiraning tashqi qismi.)

6. $1 < |z - i| < 3$ shartni qanoatlantiruvchi nuqtalarning geometrik o‘rni qanday sohani ifodalaydi? (Javob: Markazi $z = i$ nuqtada bo‘lgan R_1=1 va R_2=3 radiusli aylanalar orasidagi halqa.)

7. $0 < \mathrm{Re}(3iz) < 2$ shartni qanoatlantiruvchi nuqtalarning geometrik o‘rni qanday sohani ifodalaydi? (Javob: $y = 0$ va $y = -\frac{2}{3}$ to‘g‘ri chiziqlar orasidagi gorizontal polosa.)

8. $(1-\sqrt{3}i)^5$ ni hisoblang.(Javob: $16+16i\sqrt{3}$.)

9. $z^3 + 1 = 0$ tenglamaning ildizlarini toping. (Javob: $z_0 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $z_1 = -1$, $z_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$.)

10. Hisoblang: 1) $\sqrt[4]{-8-8i\sqrt{3}}$, 2) $\sqrt[3]{-2+2i}$..

11. Tenglamalarni yeching: 1) $z^3 - 8 = 0$, 2) $z^6 + 64 = 0$.

12. Eyler formulasidan foydalanib

$$cosx + \cos 2x + \cos 3x + ... + \cos nx$$

yig‘indini hisoblang. (Javob: $\left(\sin\frac{nx}{2}\cos\frac{(n+1)x}{2}\right)\Big/\sin\frac{x}{2}$.)

Mustaqil yechish uchun testlar

1. $z_1 = 1+3i$ va $z_2 = 3-2i$ uchun $2z_1 + z_2$ ni hisoblang.
   A) $5+4i$, B) $5-8i$, C) $5-4i$, D) $2+3i$.
2. $z_1 = 2-3i$ va $z_2 = 3-4i$ berilgan bo‘lsa, $z_1 z_2$ ni hisoblang.
   A) $18-17i$, B) $-6-i$, C) $-6-17i$, D) $6+17i$.
3. $z_1 = 3-4i$ va $z_2 = 2-i$ berilgan bo‘lsa, z_1/z_2 ni hisoblang.
   A) $3-2i$, B) $-2-i$, C) $2-i$, D) $0{,}4-i$.
4. $z = 2-2i\sqrt{3}$ kompleks sonning modulini toping.
   A) $r=3$; B) $r=4$; C) $r=5$; D) $r=1$.
5. $z = -2+2i\sqrt{3}$ kompleks sonning argumentini toping.
   A) $\varphi = \frac{\pi}{6}$; B) $\varphi = \frac{\pi}{3}$; C) $\varphi = \frac{5\pi}{6}$; D) $\varphi = \frac{2\pi}{3}$.
6. $z = 2\sqrt{3}-2i$ kompleks sonning trigonometrik shaklini toping.
   A) $4(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6})$; B) $4(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6})$;
   C) $4(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$; D) $4(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$.
7. $z = 2\sqrt{3}+2i$ kompleks sonning ko‘rsatkichli shaklini toping.
   A) $z = 4e^{i\frac{\pi}{3}}$; B) $z = 4e^{i\frac{\pi}{6}}$; C) $z = 4e^{i\frac{5\pi}{6}}$; D) $z = 4e^{-i\frac{\pi}{3}}$.
8. $z = -\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$ kompleks sonning ko‘rsatkichli shaklini toping.
   A) $z = e^{i\frac{6\pi}{7}}$; B) $z = e^{i\frac{\pi}{7}}$; C) $z = e^{-i\frac{\pi}{7}}$; D) $z = e^{-i\frac{6\pi}{7}}$.
9. $\left(\sqrt{3}-i\right)^4$ ni hisoblang.
   A) $8+8i\sqrt{3}$, B) $8-8i\sqrt{3}$, C) $-8+8i\sqrt{3}$, D) $-8-8i\sqrt{3}$.
10. $z^3+8=0$ tenglamaning yechimi noto‘g‘ri berilgan javobni aniqlang.
    A) $-1+i\sqrt{3}$, B) $1+i\sqrt{3}$, C) -2, D) $1-i\sqrt{3}$.

4.2 Funksiya va uning berilish usullari

Agar ixtiyoriy $x \in D$ elementga biror f qoida bilan yagona y element mos qo‘yilgan bo‘lsa, u holda $y = f(x)$ funksiya berilgan deyiladi. x - *erkli o‘zgaruvchi* yoki *argument* deyiladi. D - *aniqlanish soha*, y ning qabul qiladigan qiymatlari esa *qiymatlar to‘plami* (yoki *o‘z garish sohasi*) deyiladi va E harfi bilan belgilanadi.

Funksiya *jadval usulda*, *grafik usulda* va *analitik usulda* beriladi. Analitik usulda berilgan $y = f(x)$ funksiyaning D va E sohalari ko‘p hollarda ko‘rsatilmaydi, ammo tabiiy ravishda $y = f(x)$ funksiya xossalariga ko‘ra aniqlanadi.

1-misol

Ushbu $y=\sqrt{7-6x-x^2}$ funksiyaning aniqlanish sohasi va qiymatlar to‘plamini toping.

▶ Kvadrat funksiya $7-6x-x^2\geq 0$ da aniqlangan. Kvadrat uchhadning ildizlari $x_1=-7,\ x_2=1$. Yuqoridagi tengsizlik $-(x+7)(x-1)\geq 0$ tengsizlikga teng kuchli bo‘lib, $-7\leq x\leq 1$ yechimga ega. Funksiyaning aniqlanish sohasi $D=[-7;1]$. D sohada $0\leq 7-6x-x^2\leq 16$ bo‘lgani uchun qiymatlar to‘plami $E=[0;4]$. ◀

$u=\varphi(x)$ funksiya D to‘plamda aniqlangan bo‘lib, uning qiymatlar to‘plami G bo‘lsin. Agar $y=f(u)$ funksiya G to‘plamda aniqlangan funksiya bo‘lsa, u holda $y=f(\varphi(x)$ *murakkab funksiya* deyiladi. $y=f(\varphi(x)$ funksiya ikkita $y=f(u)$ va $u=\varphi(x)$ funksiyalarning kompozitsiyasi yoki φ funksiyaning f funksiyasi deb ataladi. Murakkab funksiya ikki yoki undan ortiq funksiyadan tuzilgan bo‘ladi.

2-misol

Quyidagi murakkab funksiyalar nechta funksiyadan tashkil topgan:

$a)\ y=\sin(x^2+1),\quad b)\ y=\ln\sin 3^x.$

▶ $a)$ $y=\sin(x^2+1)$ ikkita $y=\sin u$ va $u=x^2+1$ funksiyalardan tashkil topgan.

$b)$ $y=\ln\sin 3^x$ funksiya uchta $y=\ln u$, $u=\sin v$ va $v=3^x$ funksiyalardan tashkil topgan.◀

$y=f(x)$ *funksiyaning grafigi* deb Oxy tekisligidagi koordinatalari f qoida bilan bog‘langan $M(x;y)$ nuqtalar to‘plamiga aytiladi.

$\forall x\in D$ uchun $-x\in D$ bo‘lsin. Agar $\forall x\in D$ uchun $f(-x)=f(x)$ bo‘lsa, $y=f(x)$ juft funksiya deyiladi. Agar $\forall x\in D$ uchun $f(-x)=-f(x)$ bo‘lsa, $y=f(x)$ toq funksiya deyiladi.

Agar $y=f(x)$ funksiya D sohani E ga bir qiymatli akslantirsa, u holda x ni y orqali ifodalovchi funksiya $x=g(y)$ mavjud va u $y=f(x)$ ga *teskari funksiya* deyiladi. $x=g(y)$ funksiyaning aniqlanish sohasi E , qiymatlar to‘plami esa D ga teng. $y\equiv f(g(y))$ va $x\equiv g(f(x))$ bolgani uchun $y=f(x)$ va $x=g(y)$ funksiyalar o‘zaro teskari. O‘z aro teskari $y=f(x)$ va $x=g(y)$ funksiyalarning grafigi birinchi va uchinchi chorak bissektrisa chizig‘i $y=x$ ga nisbatan simmetrikdir.

3-misol

Ushbu $y=x^2-6x+11$ funksiyaga $(-\infty;3]$ oraliqdagi teskari funksiyani toping.

▶ Berilgan funksiyadan to‘la kvadrat ajratamiz

$$y=(x-3)^2+2.$$

Bu tenglikdan x ni topamiz

$$x=3\pm\sqrt{y-2}$$

va $x\in(-\infty;3]$ ekanini e’tiborga olgan holda tanlaymiz

$$x = 3 - \sqrt{y-2}.$$

x ni y ga almashtirib, izlangan funksiyani topamiz

$$y = 3 - \sqrt{x-2}. \blacktriangleleft$$

Auditoriya topshiriqlari

1. Quyidagi funksiyalarning aniqlanish sohasini toping:

1) $y = \sqrt{x^2 + 6x + 5}$; 2) $y = \lg(5 - 4x - x^2)$; 3) $y = \arcsin\dfrac{1-x}{3}$; 4) $y = 1/\sqrt{2^{3x} - 4}$.

(Javob: 1) $(-\infty; -1] \cup [-5;\ \infty)$; 2) $(-5;\ 1)$; 3) $[-2;\ 4]$; 4) $(2/3;\ \infty)$.)

2. Quyidagi murakkab funksiyalar nechta funksiyadan tashkil topgan:

1) $y = \lg\sin x^2$; 2) $y = 2^{|\cos x|}$; 3) $y = \sqrt[3]{arctg 3^{x^2}}$; 4) $y = \cos^3\sqrt{\arcsin e^x}$?

(Javob: 1) 3 ta; 2) 3 ta; 3) 4 ta; 4) 5 ta.)

3. $y = \begin{cases} x, \text{ agar } & x \le 0 \text{ bo`lsa}, \\ x^2, \text{ agar } & x > 0 \text{ bo`lsa} \end{cases}$ funksiyaning teskari funksiyasini toping. Berilgan funksiya va uning teskari funksiyasi grafigini yasang.

4. Quyidagi funksiyalarning grafiklarini yasang:

1) $y = \dfrac{2x+3}{x-1}$; 2) $y = |5 - 4x - x^2|$; 3) $y = 2\sin 2x$; 4) $y = |x-2| + |x+1|$.

5. Quyidagi funksiyalarning juft yoki toqligini aniqlang:

1) $y = \dfrac{2x^2+3}{x^2-1}$; 2) $y = |5 - 4x - x^3|$; 3) $y = 2\sin 2x + x$; 4) $y = |x-2| + |x+2|$

(Javob: 1) Juft ; 2) Juft ham, toq ham emas; 3) Toq; 4) Juft funksiya.)

6. Agar $f(x) = \lg x,\ \varphi(x) = \sin 2x,\ g(x) = x^2 + x$ bo‘lsa, funksiyani $y = g(f(\varphi(x)))$ toping. (Javob: $y = \lg^2 \sin 2x + \lg\sin 2x$.)

Mustaqil yechish uchun testlar

1. $y = \arccos\dfrac{2x}{x+3}$ funksiyaning aniqlanish sohasini toping.

   A) $(-1;\ 1)$; B) $[-1;\ 1]$; C) $(-\infty; -1] \cup [3;\ \infty)$; D) $[-1;\ 3]$.

2. $y = \sqrt[3]{\lg|\sin x^3|}$ murakkab funksiya nechta funksiyadan tashkil topgan?

   A) 3 ta; B) 4 ta; C) 5 ta; D) 2 ta.

3. Quyidagi funksiyalardan qaysilari juft ekanligini aniqlang:

   1) $y = x^3\sin 2x$; 2) $y = x\dfrac{e^x - 1}{e^x + 1}$; 3) $y = |5 - 4x + x^2|$; 4) $y = x^2 cos(x+1)$.

   A) 1), 3) B) 1), 2) C) 1), 4) D) 3), 4).

4. $y=x^2-4x+7$ funksiyaga $(-\infty;2]$ oraliqdagi teskari funksiyani toping.

A) $y=2+\sqrt{x-3}$; B) $y=2-\sqrt{x-3}$; C) $y=2+\sqrt{x+3}$; D) $y=3-\sqrt{x-2}$

5. $y=\dfrac{x-1}{2-3x}$ funksiyaga teskari funksiyani toping.

A) $y=\dfrac{2x+1}{3x+1}$; B) $y=\dfrac{2-3x}{x-1}$; C) $y=\dfrac{2x-1}{3x+1}$; D) $y=\dfrac{x+1}{2+3x}$.

4.3 Sonli ketma-ketlik va funksiya limiti

Natural sonlar to‘plamida aniqlangan funksiya *sonli ketma-ketlik* deyiladi. $x_n=f(n),\ n\in N$. x_n- ketma-ketlikning n- hadi uning *umumiy hadi* deb ataladi. Sonli ketma-ketlik $\{x_n\}$orqali belgilanadi.

Agar ixtiyoriy $\varepsilon>0$ son uchun shunday $N=N(\varepsilon)>0$ son mavjud bo‘lsaki, barcha $n\geq N$ lar uchun $|x_n-a|<\varepsilon$ tengsizlik bajarilsa, a o‘z garmas son $\{x_n\}$ *ketma-ketlikning limiti* deyiladi va quyidagicha belgilanadi:

$$\lim_{n\to\infty} x_n=a\,.$$

1-misol

Ushbu $x_n=\dfrac{1-2n^2}{2+4n^2}$ ketma ketlikning limiti $a=-\dfrac{1}{2}$ ekanini isbotlang.

► $\forall\varepsilon>0$ son uchun unga mos $N=N(\varepsilon)>0$ son mavjudligini ko‘rsatamiz:

$$|x_n-a|=\left|\frac{1-2n^2}{2+4n^2}+\frac{1}{2}\right|<\varepsilon \text{ yoki } \left|\frac{2-4n^2+2+4n^2}{2(2+4n^2)}\right|<\varepsilon,\quad \frac{1}{1+2n^2}<\varepsilon.$$

Bundan, $n>\sqrt{\dfrac{1}{2\varepsilon}-\dfrac{1}{2}}$, demak, $N(\varepsilon)=\left[\dfrac{1}{2\varepsilon}-\dfrac{1}{2}\right]$ deb tanlash kifoya. ◄

$y=f(x)$ funksiya $x=x_0$ nuqtaning biror atrofida aniqlangan bo‘lsin.

Agar ixtiyoriy $\varepsilon>0$ son uchun shunday $\delta=\delta(\varepsilon)>0$ son mavjud bo‘lsaki, $0<|x-x_0|<\delta$ tengsizlikni qanoatlantiruvchi barcha x larda $|f(x)-A|<\varepsilon$ tengsizlik o‘rinli bo‘lsa, A son $y=f(x)$ *funksiyaning* $x\to x_0$ *dagi limiti* deyiladi va quyidagicha belgilanadi:

$$\lim_{x\to x_0} f(x)=A\,.$$

Xuddi shu kabi

$$\lim_{\substack{x\to x_0\\ x<x_0}} f(x)=A\ \left(\lim_{\substack{x\to x_0\\ x>x_0}} f(x)=A\right)$$

limit mavjud bo‘lsa, bu limit $y = f(x)$ funksiyaning x_0 *nuqtadagi chap(o‘ng) limiti* deyiladi va $\lim\limits_{x\to x_0-0} f(x) = f(x_0-0)$ $\left(\lim\limits_{x\to x_0+0} f(x) = f(x_0+0)\right)$ kabi belgilanadi.

Agar ixtiyoriy $\varepsilon > 0$ son uchun shunday $N = N(\varepsilon) > 0$ son mavjud bo‘lsaki, $|x| > N$ tengsizlikni qanoatlantiruvchi barcha x larda $|f(x) - A| < \varepsilon$ tengsizlik o‘rinli bo‘lsa, A son $y = f(x)$ *funksiyaning* $x\to\infty$ *dagi limiti* deb ataladi va $\lim\limits_{x\to\infty} f(x) = A$ kabi belgilanadi.

$f(x)$ va $g(x)$ funksiyalar $x = x_0$ nuqtaning biror atrofida aniqlangan bo‘lib, $\lim\limits_{x\to x_0} f(x) = A$, $\lim\limits_{x\to x_0} g(x) = B$ bo‘lsin. U holda quyidagi tengliklar o‘rinli:

$$\lim_{x\to x_0}\big(f(x)+g(x)\big) = A+B;\ \lim_{x\to x_0}\big(f(x)g(x)\big) = A\cdot B \quad \text{va} \quad \lim_{x\to x_0}\frac{f(x)}{g(x)} = \frac{A}{B},\ B\neq 0$$

Limitlarni hisoblashda quyidagilardan foydalanamiz:

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \frac{a}{0} = \infty,\ \lim_{x\to\infty}\frac{f(x)}{g(x)} = \frac{\infty}{a} = \infty,\ \lim_{x\to\infty}\frac{f(x)}{g(x)} = \frac{a}{\infty} = 0,\ a - \text{chekli son.}$$

2-misol

Quyidagi limitlarni hisoblang:

$$1)\ \lim_{x\to 2}\left(\frac{4}{x^2-4} - \frac{1}{x-2}\right); \quad 2)\ \lim_{x\to\pm\infty}\frac{2x^3+x-2}{x^3+4x+5}.$$

► 1) Bu yerda $\infty - \infty$ tipidagi aniqmaslik, uni yechish uchun kasrga umumiy maxraj beriladi va sodda ko‘rinishga olib kelinadi:

$$\lim_{x\to 2}\left(\frac{4}{x^2-4} - \frac{1}{x-2}\right) = \lim_{x\to 2}\frac{2-x}{(x-2)(x+2)} = \lim_{x\to 2}\left(-\frac{1}{x+2}\right) = -\frac{1}{4}.$$

2) Bu holda ∞/∞ tipidagi aniqmaslik, uni yechish uchun kasrning surat va maxrajini x ning yuqori darajasi x^3 ga bo‘lamiz:

$$\lim_{x\to\infty}\frac{2x^3+x-2}{x^3+4x+5} = \lim_{x\to\infty}\frac{2+\frac{1}{x^2}-\frac{2}{x^3}}{1+\frac{4}{x^2}+\frac{5}{x^3}} = \frac{\lim\limits_{x\to\infty}2+\lim\limits_{x\to\infty}\frac{1}{x^2}-\lim\limits_{x\to\infty}\frac{2}{x^3}}{\lim\limits_{x\to\infty}1+\lim\limits_{x\to\infty}\frac{4}{x^2}+\lim\limits_{x\to\infty}\frac{5}{x^3}} = \frac{2+0-0}{1+0+0} = 2. \quad ◄$$

Auditoriya topshiriqlari

1. $x_n = \dfrac{3n-1}{n+2}$ ketma ketlikning limiti $a=3$ ekanini isbotlang.

Berilgan funksiyalarning limitlarini hisoblang.

2. $\lim\limits_{n\to\infty} \dfrac{2n^2-5n+2}{4-n^2}$. (Javob: -2)

3. $\lim\limits_{n\to\infty} \dfrac{\left(\sqrt{n^2+5}+2n\right)^2}{\sqrt[3]{n^6+2}}$. (Javob: 9)

4. $\lim\limits_{n\to\infty} \dfrac{(n+1)!+(n+2)!}{(n+3)!}$. (Javob: 0)

5. $\lim\limits_{x\to\frac{1}{2}} \dfrac{8x^3-1}{6x^2-5x+1}$. (Javob: 6)

6. $\lim\limits_{x\to1} \left(\dfrac{x+2}{x^2-5x+4}+\dfrac{x-4}{3\left(x^2-3x+2\right)}\right)$. (Javob:$0$)

7. $\lim\limits_{x\to2} \dfrac{\sqrt{x+2}-2}{3-\sqrt{x+7}}$. (Javob: $-\dfrac{3}{2}$)

8. $\lim\limits_{x\to+\infty} x\left(\sqrt{x^2+4}-x\right)$. (Javob: 2)

Mustaqil yechish uchun testlar

Sonli ketma-ketlik limitlarini hisoblang.

1. $\lim\limits_{n\to\infty} \dfrac{n^2-\sqrt{n^3+1}}{\sqrt[3]{n^6+2}-n}$.

A) 1; B) 2; C) 3; D) 0.

2. $\lim\limits_{n\to\infty} \left[\sqrt{n(n+2)}-\sqrt{n^2-2n+3}\right]$.

A) 1; B) 2; C) 3; D) 0.

3. $\lim\limits_{n\to\infty} \dfrac{2^{n+1}+3^{n+1}}{2^n+3^n}$.

A) 2; B) 2,5; C) 3; D) 5.

Funksiya limitlarini hisoblang.

4. $\lim\limits_{x\to2} \dfrac{x^2-7x+10}{x^2-5x+6}$.

A) 1; B) 2; C) 3; D) 24.

5. $\lim\limits_{x\to3} \dfrac{x^2-9}{\sqrt{x+1}-2}$.

A) 0; B) 12; C) 24; D) ∞.

6. $\lim\limits_{x\to+\infty} x\left(\sqrt{x^2+3}-\sqrt{x^2-1}\right)$.

A) 1; B) 2; C) 3; D) 0.

7. $\lim\limits_{x\to 1}\left(\dfrac{1}{1-x}-\dfrac{3}{1-x^3}\right)$.

A) 0; B) -2; C) 3; D) -1.

8. $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x}-1}{\sqrt{1+x}-\sqrt{2x}}$.

A) $1{,}5\sqrt{2}$; B) $2\sqrt{2}/3$; C) $-2\sqrt{2}/3$ D) $-1{,}5\sqrt{2}$

4.4 Ajoyib limitlar

Funksiyalarning limitlarini hisoblashda 1- *ajoyib limit* va 2- *ajoyib limit* deb ataluvchi

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \tag{4.1}$$

va

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\lim_{\alpha\to 0}(1+\alpha)^{\frac{1}{\alpha}}=e\approx 2{,}71828 \tag{4.2}$$

limitlar, hamda ularga asoslangan quyidagi formulalar keng qo‘llanadi:

1) $\lim\limits_{x\to 0}\dfrac{\sin\alpha x}{\sin\beta x}=\dfrac{\alpha}{\beta}$, 2) $\lim\limits_{x\to 0}\dfrac{tg\,kx}{x}=k$, 3) $\lim\limits_{x\to 0}\dfrac{\arcsin kx}{x}=k$, 4) $\lim\limits_{x\to 0}\dfrac{arctgkx}{x}=k$,

5) $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=1$, 6) $\lim\limits_{x\to 0}\dfrac{\log_a(1+x)}{x}=\log_a e$, 7) $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\ln a\,(a>0)$,

8) $\lim\limits_{x\to 0}\dfrac{(1+x)^m-1}{x}=m$, 9) $\lim\limits_{x\to\infty}\left(1\pm\dfrac{n}{x}\right)^x=e^{\pm n}$, 10) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x+n}=e$.

1-misol

Quyidagi limitlarni hisoblang:

$$1)\ \lim_{x\to 0}\frac{\sin 7x}{\sin 3x};\quad 2)\ \lim_{x\to\infty}\left(\frac{3x+2}{3x-1}\right)^{2x+1};\quad 3)\ \lim_{x\to\pi}\frac{tg\,2x}{\sin 3x}.$$

► 1) Berilgan limitni hisoblashda 1-ajoyib limitdan foydalanamiz. Buning uchun quyidagicha almashtirish bajaramiz:

$$\lim_{x\to 0}\frac{\sin 7x}{\sin 3x}=\lim_{x\to 0}\frac{\sin 7x}{7x}\cdot\frac{7x}{\dfrac{\sin 3x}{3x}\cdot 3x}=\lim_{x\to 0}\frac{\sin 7x}{7x}\cdot\frac{7}{3\lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}}=\frac{7}{3}.$$

2) Bu limit va shu kabi limitlarni hisoblashda berilgan funksiya asosiga birni qo‘shib ayriladi va 2-ajoyib limitga keltiriladi:

$$\lim_{x\to\infty}\left(\frac{3x+2}{3x-1}\right)^{2x+1}=\lim_{x\to\infty}\left(1+\frac{3x+2}{3x-1}-1\right)^{2x+1}=\lim_{x\to\infty}\left[\left(1+\frac{3}{3x-1}\right)^{\frac{3x-1}{3}}\right]^{\frac{6x+3}{3x-1}}=e^{\lim\limits_{x\to\infty}\frac{6x+3}{3x-1}}=e^2.$$

3) Bu limitni hisoblashda trigonometrik funksiyalarning davriyligidan va keltirish formulalaridan foydalanib, 1-ajoyib limitga keltiramiz:

$$\lim_{x\to\pi}\frac{tg\,2x}{\sin 3x}=\lim_{x\to\pi}\frac{tg(2x-2\pi)}{-\sin(3x-3\pi)}=-\lim_{x\to\pi}\frac{\frac{\sin(2x-2\pi)}{2x-2\pi}}{\frac{\sin(3x-3\pi)}{3x-3\pi}}\cdot\lim_{x\to\pi}\frac{2}{3\cos(2x-2\pi)}=-\frac{2}{3}.$$ ◄

Auditoriya topshiriqlari

Berilgan limitlarni hisoblang.

1. $\lim\limits_{x\to 0}\dfrac{3x^2-5x}{\sin 3x}$.(Javob: $-\dfrac{5}{3}$.)
2. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{\sin[\pi(x+2)]}$.(Javob: $1/(2\pi)$.)
3. $\lim\limits_{x\to 0}\dfrac{1-\cos^3 x}{x\sin 2x}$. (Javob: $\dfrac{3}{4}$.)
4. $\lim\limits_{x\to 1}(1-x)tg\dfrac{\pi x}{2}$. (Javob: $\dfrac{2}{\pi}$.)
5. $\lim\limits_{x\to 0}\dfrac{\sqrt{3x+4}-2}{\cos(\pi(x+1)/2)}$. (Javob: $-\dfrac{3}{2\pi}$.)
6. $\lim\limits_{x\to 0}\dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$. (Javob: $\dfrac{\sqrt{2}}{8}$.)
7. $\lim\limits_{x\to 0}\dfrac{1-\cos 2x+2tg^2 x}{x\sin 3x}$. (Javob: $\dfrac{4}{3}$.)
8. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x\sin x}-\cos x}{\sin^2 x}$. (Javob: 1.)
9. $\lim\limits_{x\to -\infty}\left(\dfrac{x-1}{2x+3}\right)^{3x}$. (Javob: ∞.)
10. $\lim\limits_{x\to -\infty}\left(\dfrac{3x-2}{3x+2}\right)^{2-x}$. (Javob: $\sqrt[3]{e^4}$.)
11. $\lim\limits_{x\to\infty}x(\ln(2x+3)-\ln(2x-1))$ (Javob:2.)
12. $\lim\limits_{x\to 0}(\ln(1+3tg^2 x))^{ctg^2 x}$.(Javob: 3.)
13. $\lim\limits_{x\to \frac{\pi}{4}}(\sin 2x)^{tg^2 2x}$.(Javob: $\dfrac{1}{\sqrt{e}}$.)
14. $\lim\limits_{x\to e}\dfrac{\ln x-1}{x-e}$.(Javob: $\dfrac{1}{e}$.)
15. $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{\sin 3x}$. (Javob: $\dfrac{2}{3}$.)
16. $\lim\limits_{x\to 0}\dfrac{\ln\cos x}{x^2}$.(Javob: $-\dfrac{1}{2}$.)

Mustaqil yechish uchun testlar

Berilgan limitlarni hisoblang.

1. $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\sqrt{x+1}-1}$.

A) 1,5; B) 2; C) 3; D) 6.

2. $\lim\limits_{x\to 0}\dfrac{1-\cos 3x}{x\sin x}$.

A) 3,5; B) 4; C) 4,5; D) 6.

3. $\lim\limits_{x\to \pi}\dfrac{1+\cos x}{\sin^2 2x}$.

A) 1,5; B) 0,25; C) 0,125; D) 0.5.

4. $\lim\limits_{x\to \pi}\dfrac{\cos\frac{x}{2}}{x-\pi}$.

A) 1,5; B) - 0,25; C) - 0,5; D) 0.5.

5. $\lim\limits_{x\to 0}\dfrac{\arcsin 2x}{\sqrt[3]{x+1}-1}$

A) 1,5; B) 2; C) 3; D) 6.

6. $\lim\limits_{x\to +\infty}\left(\dfrac{x+2}{2x-1}\right)^{x+3}$.

A) 0; B) 2; C) 0,5; D) ∞.

7. $\lim\limits_{x\to 0}(1-3x)^{\frac{x-1}{x}}$.

A) e; B) e^2; C) e^3; D) 0.

8. $\lim\limits_{x\to +\infty}\left(\dfrac{x^2+2}{x^2-x}\right)^{x}$.

A) e; B) e^2; C) e^3; D) 0.

9. $\lim\limits_{x\to +\infty}(x+1)(\ln(3x+2)-\ln(3x-1))$.

A) e; B) e^2; C) 1; D) 0.

10. $\lim\limits_{x\to 0}\dfrac{e^{\sin x}-1}{\sin 2x}$.

A) 1,5; B) 2; C) 2,5; D) 0.5.

4.5 Funksiya uzluksizligi. Uzilish turlari

Agar $y=f(x)$ funksiya $x=x_0$ nuqtaning biror atrofida aniqlangan bo‘lib, $\lim\limits_{x\to x_0} f(x)=f(x_0)$ bo‘lsa, u holda $y=f(x)$ funksiya $x=x_0$ *nuqtada uzluksiz* deyiladi. Bu ta’rif uzluksizlikning quyidagi shartlarini o‘z ichiga oladi:

1) $f(x)$ funksiya $x=x_0$ nuqtaning biror atrofida aniqlangan;

2) $\lim\limits_{x\to x_0-0} f(x),\ \lim\limits_{x\to x_0+0} f(x)$ chekli limitlar mavjud;

3) ular o‘z aro teng $\lim\limits_{x\to x_0-0} f(x)=\lim\limits_{x\to x_0+0} f(x)$;

4) bu limit $f(x)$ funksiyaning $x=x_0$ nuqtadagi qiymatiga teng.

$y=f(x)$ funksiya x_0 nuqtada uzluksiz bo‘lishi uchun argumentning cheksiz kichik orttirmasi Δx ga funksiyaning cheksiz kichik orttirmasi Δy mos kelishi zarur va yetarli, ya’ni uzluksizlik $\lim\limits_{\Delta x\to 0}\Delta y=0$ shart bajarilishiga teng kuchli.

1-misol

Berilgan $y = x^2$ funksiya ixtiyoriy $x \in \boldsymbol{R}$ nuqtada uzluksiz ekanini isbotlang.

► Argumentning ixtiyoriy Δx orttirmasida funksiya orttirmasi

$$\Delta y = (x + \Delta x)^2 - x^2 = 2x\Delta x + \Delta x^2.$$

U holda

$$\lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to 0} \left(2x\Delta x + \Delta x^2\right) = 0.$$

Bundan, $y = x^2$ funksiya butun sonlar o‘qida uzluksiz ekani kelib chiqadi.◄

x_0 nuqtada yuqoridagi shartlardan kamida bittasi bajarilmasa, x_0 nuqta $y = f(x)$ funksiyaning *uzilish nuqtasi* deyiladi. Agar x_0 nuqtada $f(x_0 - 0)$, $f(x_0 + 0)$ chekli limitlar mavjud va $f(x_0 - 0) \neq f(x_0 + 0)$ bo‘lsa, x_0 *birinchi tur uzilish nuqtasi* deyiladi. Agar $f(x_0 - 0)$ yoki $f(x_0 + 0)$ limitlardan hech bo‘lmaganda bittasi mavjud bo‘lmasa yoki cheksizlikka teng bo‘lsa, x_0 *ikkinchi tur uzilish nuqtasi* deyiladi. Agar x_0 nuqtada $f(x_0 - 0)$, $f(x_0 + 0)$ chekli limitlar mavjud va $f(x_0 - 0) = f(x_0 + 0)$ bo‘lib, x_0 nuqtada funksiya aniqlanmagan bo‘lsa, x_0 *yo‘qotish mumkin bo‘lgan uzilish nuqtasi* deyiladi.

2-misol

Ushbu $y = \dfrac{\sin x}{x}$ funksiyaning uzilish nuqtasini toping. Uzilish turini aniqlang.

► $\lim\limits_{x \to -0} \dfrac{\sin x}{x} = \lim\limits_{x \to +0} \dfrac{\sin x}{x} = 1$ bo‘lgani holda $x = 0$ da funksiya aniqlanmagan. Demak, $x = 0$ yo‘qotish mumkin bo‘lgan uzilish nuqtadir.◄

3-misol

Ushbu $y = \dfrac{x-2}{|x-2|}$ funksiyaning uzilish nuqtasini toping. Uzilish turini aniqlang.

► Berilgan funksiya $x = 2$ nuqtada aniqlanmagan. $\lim\limits_{x \to 2-0} y = -1$, $\lim\limits_{x \to 2+0} y = 1$ chekli limitlar mavjud va o‘z aro teng bo‘lmagani uchun $x = 2$ birinchi tur uzilish nuqtasi bo‘ladi.◄

Auditoriya topshiriqlari

1. $y = 2 - \dfrac{|x|}{x}$ funksiyani uzluksizlikga tekshiring va grafigini yasang.(Javob: $x = 0$ birinchi tur uzilish nuqtasi.)

2. $y = \begin{cases} x + 1, & \text{agar} \quad x \le 1 \text{ bo`lsa,} \\ 3 - ax^2, & \text{agar} \quad x > 1 \text{ bo`lsa} \end{cases}$ funksiya a ning qanday qiymatida uzluksiz bo‘ladi? (Javob: $a = 1$.)

3. $y=\begin{cases}\dfrac{x-1}{|x-1|}, & \text{agar} \quad x\neq 1 \text{ bo`lsa,}\\ 0, \ \text{agar} & x=1 \text{ bo`lsa}\end{cases}$ funksiyani uzluksizlikga tekshiring va uzilish turini aniqlang. (Javob: $x=1$ - birinchi tur uzilish nuqtasi.)

4. $x=0$ nuqta $y=\dfrac{1}{1+2^{\frac{1}{x}}}$ funksiyaning 1-tur uzilish nuqtasi ekanini isbotlang. $x=0$ nuqta atrofida grafigini chizing.

5. $y=2^{\frac{1}{3-x}}$ funksiyani $x=2$ va $x=3$ nuqtalarda uzluksizlikga tekshiring. (Javob: $x=2$ da uzluksiz, $x=3$ - ikkinchi tur uzilish nuqtasi.)

6. $y=\left(1+\dfrac{1}{x}\right)^x$ funksiyani uzluksizlikga tekshiring va uzilish turini aniqlang. (Javob: $x=-1$ - ikkinchi tur uzilish nuqtasi, $x=0$ - yo‘qotish mumkin bo‘lgan uzilish nuqtasi.)

7. $y=\dfrac{\sin x}{\pi^2-x^2}$ funksiyani uzluksizlikga tekshiring va uzilish turini aniqlang. (Javob: $x=\pi$ -yo‘qotish mumkin bo‘lgan uzilish nuqtasi.)

Mustaqil yechish uchun testlar

1. a ning qanday qiymatida $y=\begin{cases}x^2-1, & \text{agar} \quad x\leq 2 \text{ bo`lsa,}\\ 7-ax, & \text{agar} \quad x>2 \text{ bo`lsa}\end{cases}$ funksiya uzluksiz bo‘ladi?

A) 1; B) 2; C) 3; D) 4.

2. $y=arctg\,\dfrac{1}{x}$ funksiyani $x=0$ nuqtada uzluksizlikka tekshirilsin.

A) Uzluksiz; B) 1-tur uzilish; C) 2-tur uzilish; D) Yo‘qotish mumkin bo‘lgan uzilish.

3. a va b ning qanday qiymatlarida $y=\begin{cases}x-1, & \text{agar} \quad x\leq -1 \text{ bo`lsa}\\ ax^2+b, & \text{agar} \quad -1<x<2 \text{ bo`lsa}\\ 5-2x, & \text{agar} \quad x\geq 2 \text{ bo`lsa}\end{cases}$ funksiya uzluksiz bo‘ladi?

A) -4 va 2; B) 2 va -4; C) 1 va -3; D) -3 va 1.

4. $y=\begin{cases}3x+5, & \text{agar} \quad x\leq -1 \text{ bo`lsa,}\\ 1-x, & \text{agar} \quad -1<x\leq 1 \text{ bo`lsa,}\\ \ln x, & \text{agar} \quad x>1 \text{ bo`lsa}\end{cases}$ funksiyaning uzilish nuqtalarini toping.

A) $x=1$; B) $x=\pm 1$; C) $x=-1$; D) funksiya uzluksiz.

5. $y=1+2^{\frac{1}{x}}$ funksiyani $x=0$ nuqtada uzluksizlikga tekshiring.

A) Uzluksiz; B) 1-tur uzilish; C) 2-tur uzilish; D) Yo‘qotish mumkin bo‘lgan uzilish.

6. $y=\dfrac{1}{\lg|x|}$ funksiyaning nechta uzilish nuqtasi mavjud?

A) 0; B) 1; C) 2; D) 3.

Shaxsiy uy topshiriqlari

Berilgan limitlarni hisoblang.

1

1.1. $\lim\limits_{x\to 2}\dfrac{x^2-7x+10}{2x^2+3x-14}$.

1.2. $\lim\limits_{x\to -2}\dfrac{3x^2+7x+2}{x^2+5x+6}$.

1.3. $\lim\limits_{x\to 3}\dfrac{x^3-7x-6}{3x^2-5x-12}$.

1.4. $\lim\limits_{x\to 1}\dfrac{2x^2-x-1}{3x^2-2x-1}$.

1.4. $\lim\limits_{x\to 1/3}\dfrac{3x^2+2x-1}{27x^3-1}$.

1.5. $\lim\limits_{x\to -3}\dfrac{12+x-x^2}{x^3+27}$.

1.6. $\lim\limits_{x\to 2}\dfrac{2x^2-x-6}{2x^3-5x^2+4}$.

1.7. $\lim\limits_{x\to -4}\dfrac{x^2+3x-4}{2x^2-9x-14}$.

1.8. $\lim\limits_{x\to 1/2}\dfrac{4x^2+4x-3}{2x^2+x-1}$.

1.9. $\lim\limits_{x\to -1}\dfrac{3x^2-x-4}{1-4x^2-3x^3}$.

1.10. $\lim\limits_{x\to 5}\dfrac{x^2-2x-15}{x^2-10x+25}$.

1.11. $\lim\limits_{x\to -1}\dfrac{5x^2+4x-1}{3x^3-2x+1}$.

1.12. $\lim\limits_{x\to 7}\dfrac{x^2-8x+7}{x^2-5x-14}$.

1.13. $\lim\limits_{x\to -8}\dfrac{2x^2+15x-8}{3x^2+25x+8}$.

1.14. $\lim\limits_{x\to 2}\dfrac{-5x^2+11x-2}{3x^2-x-10}$.

1.15. $\lim\limits_{x\to 1}\dfrac{7x^2-4x-3}{3x^3-2x-1}$.

1.16. $\lim\limits_{x\to 4}\dfrac{x^2-x-12}{3x^2-3x-12}$.

1.17. $\lim\limits_{x\to 5}\dfrac{x^2-3x-10}{2x^2-13x+15}$.

1.18. $\lim\limits_{x\to -2}\dfrac{2x^2+3x-2}{3x^3-13x-2}$.

1.19. $\lim\limits_{x\to -2}\dfrac{4x^2+7x-2}{3x^2+8x+4}$.

1.20. $\lim\limits_{x\to -3}\dfrac{x^3-7x+6}{3x^2+7x-6}$.

1.21. $\lim\limits_{x\to 4}\dfrac{x^2-3x-4}{3x^2-2x-40}$.

1.22. $\lim\limits_{x\to -3}\dfrac{2x^2+5x-3}{3x^2+10x+3}$.

1.23. $\lim\limits_{x\to 1}\dfrac{4x^2-5x+1}{x^3-8x+7}$.

1.24. $\lim\limits_{x\to 1/4}\dfrac{12x^2+13x-4}{4x^2-5x+1}$.

1.25. $\lim\limits_{x\to 2}\dfrac{2x^2+5x-18}{3x^3-5x^2-4}$.

1.26. $\lim\limits_{x\to -2}\dfrac{x^2+4x+4}{3x^2+5x-2}$.

1.27. $\lim\limits_{x\to 1/3}\dfrac{27x^3-1}{3x^2+5x-2}$.

1.28. $\lim\limits_{x\to -5}\dfrac{2x^2+7x-15}{x^2+10x+25}$.

1.29. $\lim\limits_{x\to 1}\dfrac{2x^3-x-1}{5x^2-2x-3}$.

1.30. $\lim\limits_{x\to 5}\dfrac{x^2-9x+20}{x^2-x-20}$

2

2.1. $\lim\limits_{x\to\infty}\dfrac{2x^2-x-1}{3x^2-2x-1}$.

2.2. $\lim\limits_{x\to\infty}\dfrac{2x^3-5x-1}{5x^3-2x^2+3}$.

2.3. $\lim\limits_{x\to\infty}\dfrac{3x^2+5x-2}{2x^2-3x+1}$.

2.4. $\lim\limits_{x\to\infty}\dfrac{2x^2-3x-5}{2+5x-3x^2}$.

2.5. $\lim\limits_{x\to\infty}\dfrac{7x^3+4x}{x^3-3x+2}$.

2.6. $\lim\limits_{x\to\infty}\dfrac{3x^4+2x-5}{2x^3+x+7}$.

2.7. $\lim\limits_{x\to\infty}\dfrac{4-3x-2x^2}{x^2+12x+13}$.

2.8. $\lim\limits_{x\to\infty}\dfrac{5x^3-x^2-1}{3x^2-2x^3}$.

2.9. $\lim\limits_{x\to\infty}\dfrac{2x-x^3}{3x^3+2x^2+1}$.

2.10. $\lim\limits_{x\to\infty}\dfrac{4x^3-2x+1}{2x^2-5x-3}$.

2.11. $\lim\limits_{x\to\infty}\dfrac{2x^3+3x^2-1}{5-3x^3}$.

2.12. $\lim\limits_{x\to\infty}\dfrac{5x^3-7x^2+3}{2+2x-x^2}$.

2.13. $\lim\limits_{x\to\infty}\dfrac{2x^3+3x^2-5x}{5+3x^2-2x^3}$.

2.14. $\lim\limits_{x\to\infty}\dfrac{2x^2-x-1}{3x^3-2x+5}$.

2.15. $\lim\limits_{x\to\infty}\dfrac{x^3-7x^2+1}{3x^2-2x+4}$.

2.16. $\lim\limits_{x\to\infty}\dfrac{5+3x-4x^2}{x^2+2x+3}$.

2.17. $\lim\limits_{x\to\infty}\dfrac{4-3x^2}{x^3+5x-6}$.

2.18. $\lim\limits_{x\to\infty}\dfrac{7-2x^2}{2x^3+5x-7}$.

2.19. $\lim\limits_{x\to\infty}\dfrac{4-5x^2-3x^5}{x^5+6x+8}$.

2.20. $\lim\limits_{x\to\infty}\dfrac{2x^4+1}{8x^4+5x^2+13}$.

2.21. $\lim\limits_{x\to\infty}\dfrac{5-2x-3x^2}{3x^3+2x-5}$.

2.22. $\lim\limits_{x\to\infty}\dfrac{4x^3+3x-7}{1-2x^3}$.

2.23. $\lim\limits_{x\to\infty}\dfrac{4-3x-2x^2}{x^2+12x+13}$.

2.24. $\lim\limits_{x\to\infty}\dfrac{3-2x-2x^2}{2x^2+5x+3}$.

2.25. $\lim\limits_{x\to\infty}\dfrac{2x^2+8x-11}{-x^2+3x+4}$.

2.26. $\lim\limits_{x\to\infty}\dfrac{2x^4+5}{x^3+2x^2+3}$.

2.27. $\lim\limits_{x\to\infty}\dfrac{3x^2-5x+7}{3x^2+x+1}$.

2.28. $\lim\limits_{x\to\infty}\dfrac{3x^3-2x+5}{6x^2+5x+1}$.

2.29. $\lim\limits_{x\to\infty}\dfrac{4-x^2}{x^2+2x+1}$.

2.30. $\lim\limits_{x\to\infty}\dfrac{1+3x-2x^3}{4x^2+2x-6}$.

3

3.1. $\lim\limits_{x\to -3}\dfrac{6-x-x^2}{\sqrt{x+7}-2}$.

3.2. $\lim\limits_{x\to 3}\dfrac{2x^2-5x-3}{\sqrt{x+2}-\sqrt{8-x}}$.

3.3. $\lim\limits_{x\to-1}\dfrac{3-\sqrt{10+x}}{3x^2+2x-1}$.

3.4. $\lim\limits_{x\to-2}\dfrac{\sqrt{6+x}-2}{3x^2+4x-4}$.

3.5. $\lim\limits_{x\to3}\dfrac{\sqrt{6+x}-3}{x^3-27}$.

3.6. $\lim\limits_{x\to0}\dfrac{5x}{\sqrt{5+x}-\sqrt{5-x}}$.

3.7. $\lim\limits_{x\to-2}\dfrac{\sqrt{7+x}-\sqrt{3-x}}{2x^2+5x+2}$.

3.8. $\lim\limits_{x\to4}\dfrac{2x^2-9x+4}{\sqrt{x-3}-\sqrt{5-x}}$.

3.9. $\lim\limits_{x\to0}\dfrac{\sqrt{7x}}{\sqrt{x+7}-\sqrt{7-x}}$.

3.10. $\lim\limits_{x\to4}\dfrac{x^2-4x}{\sqrt{2x+1}-\sqrt{x+5}}$.

3.11. $\lim\limits_{x\to0}\dfrac{x^2+5x}{\sqrt{2x+3}-\sqrt{3-x}}$.

3.12. $\lim\limits_{x\to0}\dfrac{2-\sqrt{x^2+4}}{x^2}$.

3.13. $\lim\limits_{x\to5}\dfrac{2x^2-7x-15}{\sqrt{x+4}-3}$.

3.14. $\lim\limits_{x\to3}\dfrac{x^2-5x+6}{\sqrt{5x+1}-4}$.

3.15. $\lim\limits_{x\to-4}\dfrac{x^3+64}{4-\sqrt{20+x}}$.

3.16. $\lim\limits_{x\to0}\dfrac{x^2+2x}{\sqrt{x+9}-3}$.

3.17. $\lim\limits_{x\to5}\dfrac{5-\sqrt{4x+5}}{x^2-3x-10}$.

3.18. $\lim\limits_{x\to1}\dfrac{\sqrt{2+7x}-3}{3-\sqrt{x+8}}$.

3.19. $\lim\limits_{x\to-2}\dfrac{\sqrt{x+7}-\sqrt{5}}{2-5x-3x^2}$.

3.20. $\lim\limits_{x\to3}\dfrac{x^3-27}{3-\sqrt{x+6}}$.

3.21. $\lim\limits_{x\to-3}\dfrac{x^2+5x+6}{\sqrt{x+4}-\sqrt{7+2x}}$.

3.22. $\lim\limits_{x\to4}\dfrac{\sqrt{3x+4}-4}{x^3-16x}$.

3.23. $\lim\limits_{x\to0}\dfrac{\sqrt{x^2+4}-2}{x^3-16x}$.

3.24. $\lim\limits_{x\to4}\dfrac{\sqrt{4x}-x}{x^2-4x}$.

3.25. $\lim\limits_{x\to2}\dfrac{\sqrt{3x-2}-x}{3x^2-x-10}$.

3.26. $\lim\limits_{x\to-2}\dfrac{\sqrt{4+x}-\sqrt{2}}{3x^2+5x-2}$.

3.27. $\lim\limits_{x\to-4}\dfrac{\sqrt{12-x}+x}{x^3+4x^2}$.

3.28. $\lim\limits_{x\to2}\dfrac{\sqrt{3+x}-\sqrt{7-x}}{x^3-6x+4}$.

3.29. $\lim\limits_{x\to9}\dfrac{\sqrt{2x+7}-5}{3-\sqrt{x}}$.

3.30. $\lim\limits_{x\to3}\dfrac{x^3-27}{\sqrt{3x}-x}$.

4

4.1. $\lim\limits_{x\to0}\dfrac{1-\sqrt{3x+1}}{\cos(\pi(x+1)/2)}$.

4.2. $\lim\limits_{x\to0}\dfrac{1-\cos 10x}{5x^2}$.

4.3. $\lim\limits_{x\to0}\dfrac{3x^2-5x}{\sin 3x}$.

4.4. $\lim\limits_{x \to 0} \dfrac{1-\cos 2x}{\cos 7x - \cos 3x}$.

4.5. $\lim\limits_{x \to 0} \dfrac{4x}{\operatorname{tg}(\pi(2+x))}$.

4.6. $\lim\limits_{x \to 0} \dfrac{2x}{\operatorname{tg}[2\pi(x+1/2)]}$.

4.7. $\lim\limits_{x \to 0} \dfrac{1-\cos^3 x}{4x^2}$.

4.8. $\lim\limits_{x \to 0} \dfrac{\arcsin 3x}{\sqrt{2+x}-\sqrt{2}}$.

4.9. $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x}-1}{\sin[\pi(x+2)]}$.

4.10. $\lim\limits_{x \to 0} \dfrac{arctg 2x}{\sin(2\pi(x+10))}$.

4.11. $\lim\limits_{x \to 0} \dfrac{x \sin(\pi(x+5))}{1-\cos 2x}$.

4.12. $\lim\limits_{x \to 0} \dfrac{\cos(x+5\pi/2) tgx}{\arcsin 2x^2}$.

4.13. $\lim\limits_{x \to 0} \dfrac{2x \sin x}{1-\cos x}$.

4.14. $\lim\limits_{x \to 0} \dfrac{x \sin 5x}{1-\cos 4x}$.

4.15. $\lim\limits_{x \to 0} \dfrac{\sin 7x}{x^2+\pi x}$.

4.16. $\lim\limits_{x \to 0} \dfrac{\cos 2x - \cos x}{1-\cos x}$.

4.17. $\lim\limits_{x \to -2} \dfrac{\operatorname{tg} \pi x}{x+2}$.

4.18. $\lim\limits_{x \to 0} \dfrac{\sqrt{4+x}-2}{3arctgx}$.

4.19. $\lim\limits_{x \to 2} \dfrac{\sin 7\pi x}{\sin 8\pi x}$.

4.20. $\lim\limits_{x \to 0} \dfrac{1-\sqrt{\cos x}}{x \sin x}$.

4.21. $\lim\limits_{x \to \pi/3} \dfrac{1-2\cos x}{\pi - 3x}$.

4.22. $\lim\limits_{x \to \pi} \dfrac{1+\cos 3x}{\sin^2 7x}$.

4.23. $\lim\limits_{x \to 0} \dfrac{\sin^2 x - tg^2 x}{x^4}$.

4.24. $\lim\limits_{x \to 1} \dfrac{3-\sqrt{10-x}}{\sin 3\pi x}$.

4.25. $\lim\limits_{x \to 0} \dfrac{tgx - \sin x}{x(1-\cos 2x)}$.

4.26. $\lim\limits_{x \to 1} \dfrac{1+\cos \pi x}{\mathrm{tg}^2 \pi x}$.

4.27. $\lim\limits_{x \to \pi/4} \dfrac{1-\sin 2x}{(\pi - 4x)^2}$.

4.28. $\lim\limits_{x \to \pi} \dfrac{1-\sin(x/2)}{\pi - x}$.

4.29. $\lim\limits_{x \to 2} \dfrac{\mathrm{arctg}(x^2 - 2x)}{\sin 3\pi x}$.

4.30. $\lim\limits_{x \to \pi} \dfrac{\cos 5x - \cos 3x}{\sin^2 x}$.

5

5.1. $\lim\limits_{x\to 0} \left(1+3tg^2 x\right)^{ctg^2 x}$.

5.2. $\lim\limits_{x\to\infty} \left(\dfrac{2x}{2x+3}\right)^{-4x}$.

5.3. $\lim\limits_{x\to\infty} \left(x \ln \dfrac{x}{x+3}\right)$.

5.4. $\lim\limits_{x \to \pi/2} \left(\mathrm{tg}\dfrac{x}{2}\right)^{1/(x-\pi/2)}$.

5.5. $\lim\limits_{x\to\infty} \left(\dfrac{2x-3}{2x+5}\right)^{5x-2}$.

5.6. $\lim\limits_{x\to 0}\dfrac{\ln(1+\sin^2 2x)}{5x^2}$.

5.7. $\lim\limits_{x\to 3}\left(\dfrac{9-2x}{3}\right)^{\mathrm{tg}\frac{\pi x}{6}}$.

5.8. $\lim\limits_{x\to\infty}\left(\dfrac{3-2x}{1-2x}\right)^{3x-2}$.

5.9. $\lim\limits_{x\to 0}\left(5-\dfrac{4}{\cos x}\right)^{1/\sin^2 3x}$.

5.10. $\lim\limits_{x\to 0}\left(\mathrm{tg}\left(\dfrac{\pi}{4}-x\right)\right)^{\mathrm{ctg}x}$.

5.11. $\lim\limits_{x\to\pi/2}(\sin 2x)^{tg^2 2x}$.

5.12. $\lim\limits_{x\to\infty}\left(\dfrac{x+4}{x+8}\right)^{-3x}$.

5.13. $\lim\limits_{x\to 0}\dfrac{2^x-3^x}{\sin x}$.

5.14. $\lim\limits_{x\to 0}\dfrac{\ln(1+4x^2)}{3x^2}$.

5.15. $\lim\limits_{x\to 0}\dfrac{tg\,2x}{\ln(1+3x)}$.

5.16. $\lim\limits_{x\to 1}\left(\dfrac{x-2}{2x-3}\right)^{ctg\,\pi x}$.

5.17. $\lim\limits_{x\to\pi}\dfrac{\ln(1+\sin 2x)}{x^2-\pi^2}$.

5.18. $\lim\limits_{x\to 0}\dfrac{\ln(1+\sin 3x)}{x^2+5x}$.

5.19. $\lim\limits_{x\to 0}\dfrac{\ln(\cos 2x)}{\sin^2 x}$.

5.20. $\lim\limits_{x\to\pi}\dfrac{\ln(\cos 2x)}{(x-\pi)\sin x}$.

5.21. $\lim\limits_{x\to 4}(2x-7)^{\frac{3x}{x^2-16}}$.

5.22. $\lim\limits_{x\to 1}\dfrac{\ln(1+\sin\pi x)}{x^2+2x-3}$.

5.23. $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{\ln(1+3x^2)}$.

5.24. $\lim\limits_{x\to 2}(2x-3)^{\frac{3x}{x-2}}$.

5.25. $\lim\limits_{x\to 3}\left(\dfrac{x-2}{2x-5}\right)^{2/(3-x)}$.

5.26. $\lim\limits_{x\to 0}\left(6-\dfrac{5}{\cos x}\right)^{\operatorname{ctg}^2 x}$.

5.27. $\lim\limits_{x\to 2}\left(\dfrac{4x-7}{5}\right)^{tg\frac{\pi x}{4}}$

5.28. $\lim\limits_{x\to\infty} x\left(\ln(2x+1)-\ln(2x+5)\right)$.

5.29. $\lim\limits_{x\to\infty}\left(\dfrac{x^2+2x-3}{x^2+3x+1}\right)^{2x-3}$.

5.30. $\lim\limits_{x\to\infty} x\left(\ln(3x-1)-\ln(3x-5)\right)$.

6

Berilgan funksiyalarni uzluksizlikka tekshiring, uzluksizlik oraliqlarini va uzilish nuqtalarining turini aniqlang.

6.1. $a)\ f(x)=\begin{cases} x+4, & x\le -2,\\ x^2-2, & -2<x\le 2,\\ 3x-5, & x>2,\end{cases}$ $\quad b)\ f(x)=2^{\frac{1}{x+3}}+1.$

6.2. $a)\ f(x)=\begin{cases} 0, & x<-1,\\ x^2+1, & -1\le x\le 2,\\ 3x, & x>2,\end{cases}$ $\quad b)\ f(x)=3^{\frac{1}{x-2}}+2.$

6.3. $a)\ f(x)=\begin{cases} 3x, & x\le -2,\\ x^2+2, & -2<x\le 2,\\ x+3, & x>2,\end{cases}$ $\quad b)\ f(x)=\dfrac{3}{2^x-1}.$

6.4. $a)\ f(x)=\begin{cases} 0, & x\le 0,\\ \sin x, & 0<x\le \pi,\\ 2x-6, & x>\pi,\end{cases}$ $\quad b)\ f(x)=\dfrac{1}{2^{\frac{1}{x+3}}+1}.$

6.5. $a)\ f(x)=\begin{cases}\cos x,\ x\le -\pi/2,\\ 0,\ -\pi/2<x\le \pi/2,\\ 1,\ x>\pi/2,\end{cases}$ $\qquad b)\ f(x)=4^{\frac{1}{4-x}}-1.$

6.6. $a)\ f(x)=\begin{cases}-x+1,\ x<-2,\\ x^2-1,\ -2\le x\le 2,\\ 3x-2,\ x>2,\end{cases}$ $\qquad b)\ f(x)=6^{\frac{1}{x-5}}+3.$

6.7. $a)\ f(x)=\begin{cases}1,\ x<-1,\\ -x^2+1,\ -1\le x\le 2,\\ x-5,\ x>2,\end{cases}$ $\qquad b)\ f(x)=5^{\frac{1}{x+2}}-1.$

6.8. $a)\ f(x)=\begin{cases}1,\ x<0,\\ 2^x+1,\ 0\le x\le 2,\\ 3x-1,\ x>2,\end{cases}$ $\qquad b)\ f(x)=\dfrac{x-3}{x+2}.$

6.9. $a)\ f(x)=\begin{cases}\sqrt{1-x},\ x<0,\\ x^2-2x,\ 0\le x\le 2,\\ x-1,\ x>2,\end{cases}$ $\qquad b)\ f(x)=3^{\frac{1}{x+1}}+1.$

6.10. $a)\ f(x)=\begin{cases}0,\ x<1,\\ \ln x,\ 1\le x\le 3,\\ x-1,\ x>3,\end{cases}$ $\qquad b)\ f(x)=6^{\frac{x+2}{x-2}}-1.$

6.11. $a)\ f(x)=\begin{cases}-x+1,\ x<-2,\\ x^3+5,\ -2\le x\le 1,\\ x+5,\ x>1,\end{cases}$ $\qquad b)\ f(x)=2arctg\,\dfrac{1}{x-2}.$

6.12. $a)\ f(x)=\begin{cases}3-2x,\ x<-2,\\ x^3+1,-2\le x\le 2,\\ 2x+5, x>2,\end{cases}$ $\qquad b)\ f(x)=\dfrac{(1+x)^3-1}{x}.$

6.13. $a)\ f(x)=\begin{cases}2x-3,\ x<-2,\\ x^3+1,\ -2\le x\le 2,\\ 2^x+1,\ x>2,\end{cases}$ $\qquad b)\ f(x)=\dfrac{1}{x}+\dfrac{1}{|x|}.$

6.14. $a)\ f(x)=\begin{cases}2^x-3,\ \ x<2,\\ \log_2 x,\ \ 2\le x\le 8,\\ 2x-15,\ \ x>8,\end{cases}$ $\qquad b)\ f(x)=\dfrac{1}{1-e^{1-x}}.$

6.15. $a)\ f(x)=\begin{cases}2^x-1, & x<0,\\ 2\sin x, & 0\le x\le\pi,\\ 2x-5, & x>\pi,\end{cases}$ $\quad b)\ f(x)=\dfrac{1}{1+e^{1/x}}.$

6.16. $a)\ f(x)=\begin{cases}x+\pi, & x<-\pi/2,\\ \sin x+1, & -\pi/2\le x\le\pi/2,\\ 2x+1, & x>\pi,\end{cases}$ $\quad b)\ f(x)=\dfrac{1}{1+3^{1/(x+1)}}.$

6.17. $a)\ f(x)=\begin{cases}x+1, & x<0,\\ 2\cos x, & 0\le x\le\pi,\\ 1-x, & x>\pi,\end{cases}$ $\quad b)\ f(x)=arctg\,3^{1/x}.$

6.18. $a)\ f(x)=\begin{cases}2x-1, & x<-1,\\ 3x^2, & -1\le x\le 1,\\ 2^x+1, & x>1,\end{cases}$ $\quad b)\ f(x)=\dfrac{3^{1/x}-1}{3^{1/x}+1}.$

6.19. $a)\ f(x)=\begin{cases}3x+2, & x<-1,\\ -2x^2+1, & -1\le x\le 1,\\ x-2, & x>1,\end{cases}$ $\quad b)\ f(x)=3^{1/(2-x)}+1.$

6.20. $a)\ f(x)=\begin{cases}2x+1, & x<-1,\\ -2x^2+1, & -1\le x\le 2,\\ x+5, & x>2,\end{cases}$ $\quad b)\ f(x)=2^{1/(x-3)}+1.$

6.21. $a)\ f(x)=\begin{cases}1-x, & x<-1,\\ -x^2+3, & -1\le x\le 2,\\ 2x-3, & x>2,\end{cases}$ $\quad b)\ f(x)=\dfrac{1}{3^{1/(2-x)}+1}.$

6.22. $a)\ f(x)=\begin{cases}x+2, & x<-2,\\ x^2-4, & -2\le x\le 3,\\ 2x-1, & x>3,\end{cases}$ $\quad b)\ f(x)=\dfrac{2}{3^{tgx}+1}.$

6.23. $a)\ f(x)=\begin{cases}2x+1, & x<-1,\\ -2x^2+1, & -1\le x\le 2,\\ 2-3x, & x>2,\end{cases}$ $\quad b)\ f(x)=\dfrac{3^{1/(2-x)}-1}{3^{1/(2-x)}+1}.$

6.24. $a)\ f(x)=\begin{cases}1-2x, & x<-1,\\ -2\cos\pi x+1, & -1\le x\le 2,\\ 2x-3, & x>2,\end{cases}$ $\quad b)\ f(x)=5^{1/(x+3)}+1.$

6.25. $a)\ f(x)=\begin{cases}x-5, & x<-2,\\ -2x^2+1, & -2\le x\le 2,\\ 2x+3, & x>2,\end{cases}$ $\quad b)\ f(x)=\dfrac{1}{3^{1/(2-x)}}+1.$

6.26. $a)\ f(x)=\begin{cases} x+1, & x<-1, \\ x^2-1, & -1\le x\le 2, \\ 4-x, & x>2, \end{cases}$ $\qquad b)\ f(x)=arctg\,\dfrac{2}{3-x}.$

6.27. $a)\ f(x)=\begin{cases} x+1, & x<-1, \\ x^2, & -1\le x\le 2, \\ \log_2 x+3, & x>2, \end{cases}$ $\qquad b)\ f(x)=\dfrac{x^2-1}{x^3-1}.$

6.28. $a)\ f(x)=\begin{cases} 2x+1, & x<0, \\ 1-2^x, & 0\le x\le 2, \\ x-5, & x>2, \end{cases}$ $\qquad b)\ f(x)=\dfrac{3^{1/x}-2}{3^{1/x}+2}.$

6.29. $a)\ f(x)=\begin{cases} \sin x+1, & x<0, \\ 1-x^2, & 0\le x\le 2, \\ x+1, & x>2, \end{cases}$ $\qquad b)\ f(x)=2arctg\,\dfrac{1}{3-x}.$

6.30. $a)\ f(x)=\begin{cases} 2x+1, & x<-1, \\ x^2-2, & -1\le x\le 2, \\ 8-3x, & x>2, \end{cases}$ $\qquad b)\ f(x)=\dfrac{1}{1+4^{tgx}}.$

V BOB DIFFERENSIAL HISOB ELEMENTLARI

5.1 Funksiya hosilasi

$y = f(x)$ funksiya $[a,b]$ kesmada aniqlangan bo‘lib, $x,\ x+\Delta x \in [a,b]$. Funksiyaning $\Delta y = f(x+\Delta x) - f(x)$ orttirmasini argument orttirmasi Δx ga nisbati $\frac{\Delta y}{\Delta x}$ ning Δx nolga intilgandagi limiti $y = f(x)$ funksiyaning x nuqtadagi *hosilasi* deyiladi. Quyidagi belgilardan biri bilan belgilanadi: $y',\ f'(x),\ \frac{dy}{dx}$. Demak,

$$y' = f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Agar bu limit mavjud bo‘lsa, $y = f(x)$ funksiya x nuqtada *differensiallanuvchi,* hosilani topish jarayoni esa *differensiallash* deyiladi.

$y = f(x)$ funksiyaning x nuqtadagi hosilasi funksiya grafigiga $M(x, f(x))$ nuqtasida o‘ tkazilgan urinmaning burchak koeffitsientiga teng.

Fizik nuqtai nazardan $y' = f'(x)$ hosila funksiyaning x nuqtadagi argument x ga nisbatan o‘z garish tezligini aniqlaydi.

Agar $C-$ o‘zgarmas son, $u(x)$ va $v(x)-$ differensiallanuvchi funksiyalar bo‘lsa, quyidagi *differensiallash qoidalari* o‘rinli:

$$1)\ C' = 0; \quad 2)\ (u \pm v)' = u' \pm v'; \quad 3)\ (Cu)' = Cu'; \quad 4)\ (uv)' = u'v + uv';$$

$$5)\ \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}; \quad 6)\ \left(\frac{C}{v}\right)' = -\frac{Cv'}{v^2};$$

7) agar $y = f(u)$, $u = \varphi(x)$, ya’ni $y = f(\varphi(x)) -$ differensiallanuvchi funksiyalardan tashkil topgan murakkab funksiya bo‘lsa, u holda

$$y'_x = y'_u u'_x \ \text{yoki}\ \ y' = f'(u)\varphi'(x);$$

8) agar $y = f(x)$ funksiya uchun differensiallanuvchi $x = g(y)$ teskari funksiya mavjud va $g'(y) \neq 0$ bo‘lsa, u holda

$$y'_x = \frac{1}{x'_y} \ \text{yoki}\ f'(x) = \frac{1}{g'(x)};$$

1-misol

Ushbu $y = x^3$ funksiya hosilasini ta’rif bo‘ yicha toping.

► Argumentning ixtiyoriy Δx orttirmasida

$$\Delta y = (x+\Delta x)^3 - x^3 = 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3,$$

u holda

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3}{\Delta x} = \lim_{\Delta x \to 0}\left(3x^2 + 3x\Delta x + \Delta x^2\right) = 3x^2.$$ ◄

Funksiya x nuqtada differensiallanuvchi bo‘lsa, u shu nuqtada uzluksiz bo‘ladi, aksinchasi har doim ham o‘rinli emas, ya’ni x nuqtada uzluksiz funksiya shu nuqtada differensiallanuvchi bo‘lmasligi ham mumkin.

2-misol

Ushbu $y=|x|$ funksiya $x=0$ nuqtada differensiallanuvchi bo‘ladimi?

► Funksiya berilgan nuqtada uzluksiz. Argumentning $x=0$ nuqtadagi ixtiyoriy Δx orttirmasida funksiya orttirmasi

$$\Delta y=\begin{cases}-\Delta x, \text{ agar } \Delta x<0 \text{ bo`lsa},\\ \Delta x, \text{ agar } \Delta x>0 \text{ bo`lsa}.\end{cases}$$

Hosila ta’rifiga ko‘ra,

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\begin{cases}-1, \text{ agar } \Delta x<0 \text{ bo`lsa},\\ 1, \text{ agar } \Delta x>0 \text{ bo`lsa}.\end{cases}$$

Bundan kelib chiqadiki, $y=|x|$ funksiya $x=0$ nuqtada hosilaga ega emas.◄

3-misol

Ushbu $y=\ln^3(\arcsin\sqrt{x})$ funksiyaning hosilasini toping

► Avval $y=u^3$ murakkab funksiyadan hosila hisoblaymiz $y'=3u^2u'$, bu yerda $u=\ln(\arcsin\sqrt{x})$, hamda $u=\ln v$ bo‘lgani uchun, $u'=\frac{1}{v}v'$, $v=\arcsin\sqrt{x}$. O‘z navbatida

$$v=\arcsin w,\quad v'=\frac{1}{\sqrt{1-w^2}}w',\quad w=\sqrt{x},\quad w'=(\sqrt{x})'=\frac{1}{2\sqrt{x}}.$$

Demak,

$$y'=3\ln^2(\arcsin\sqrt{x})\frac{1}{\arcsin\sqrt{x}}\cdot\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}.◄$$

Auditoriya topshiriqlari

1. $y=\dfrac{3x-2}{1+4x}$ funksiya hosilasini ta’rifdan foydalanib toping.

2. $y=\sqrt[3]{x}$ funksiya $x=0$ nuqtada uzluksiz va differensiallanuvchi bo‘ladimi?

3. Quyidagi funksiyalarning hosilalarini toping:

$a)$ $y=3x^3-4\sqrt[5]{x^4}+5/x^2$; $\quad b)$ $y=x^2\cos x\cdot\ln x$;

$c)$ $y=(x^3+2^x)tgx$; $\quad d)$ $y=e^x/(x^2+1)$

4. Hosilalar jadvali va differensiallash qoidalaridan foydalanib quyidagi funksiyalarning hosilalarini hisoblang:

$a)$ $y=arctg\sqrt{1+e^{-x}}$; $\quad b)$ $y=(2^{\cos 3x}+\sin 3x)^2$;

$c)$ $y=x^3tg\sqrt[5]{x}+3^{\sin x}$; $\quad d)$ $y=\lg^2(x^5+\sin^2 x)$;

5. $y = e^{tg\frac{x}{2}}$ funksiya $y'\sin x = y\ln y$ tenglamani qanoatlantirishini tekshiring.

Mustaqil yechish uchun testlar

Quyidagi funksiyalarning hosilalarini toping:

1. $y = 6\sqrt[3]{x} - \frac{1}{2x^2} + \frac{1}{\sqrt[3]{x^2}}$;

A) $2\sqrt[3]{x^2} + \frac{1}{x} - \frac{2}{3\sqrt{x}}$; B) $\frac{2}{\sqrt[3]{x^2}} + \frac{1}{x^3} - \frac{2}{3\sqrt[3]{x^5}}$; C) $2\sqrt[3]{x^2} - \frac{1}{x^3} + \frac{2}{3\sqrt[3]{x^5}}$; D) $\frac{2}{\sqrt[3]{x^2}} + \frac{1}{x} - \frac{2}{3\sqrt[3]{x^5}}$.

2. $y = \frac{2x+3}{x^2+1}$;

A) $\frac{2(x^2-3x+1)}{(x^2+1)^2}$; B) $\frac{2(x^2+3x-1)}{(x^2+1)^2}$; C) $\frac{2(3x^2+3x+1)}{(x^2+1)^2}$; D) $\frac{2(1-3x-x^2)}{(x^2+1)^2}$.

3. $y = (2x^2-7)\ln(x+1) + \sqrt{a}$;

A) $4x\ln(x+1) + \frac{2x^2-7}{x+1}$; B) $\frac{4x}{x-1}$; C) $4x + \frac{2x^2-7}{x+1}$; D) $4x\ln(x+1) + \frac{2x^2-7}{x+1} + \frac{1}{2\sqrt{a}}$.

4. $y = x^2\cos 3x + arctg\sqrt{x-1}$;

A) $-6x\sin 3x + \frac{1}{x\sqrt{x-1}}$; B) $2x\cos 3x - 3x^2\sin 3x + \frac{1}{2\sqrt{x^2-x}}$;

C) $-6x\sin 3x + \frac{1}{2x\sqrt{x-1}}$; D) $2x\cos 3x - 3x^2\sin 3x + \frac{1}{2x\sqrt{x-1}}$.

5. $y = \log_2 \sin^2 2^x$;

A) $\frac{2\cos 2^x}{\sin^2 2^x}\ln 2$; B) $\frac{2^x ctg\, 2^x}{\ln 2}$; C) $2^{x+1} ctg\, 2^x$; D) $\frac{2^x ctg\, 2^x}{\sin 2^x}$.

5.2 Logarifmlab differensiallash. Oshkormas va parametrik funksiya hosilalari

Funksiyani ketma-ket logarifmlash va differensiallash jarayoniga *logarifmlab differensiallash* deyiladi: $(\ln f(x))' = f'(x)/f(x)$. Bu qoida funksiyani avval logarifmlash hosila topishni soddalashtiradigan hollarda qo‘llanadi.

1-misol

Ushbu $y = \frac{(x+3)^2\sqrt[3]{x-1}}{\sqrt[4]{(x+2)^3}}$ funksiya hosilasini toping.

► Avval logarifmlash maqsadga muvofiq,

$$\ln y = 2\ln(x+3) + \frac{1}{3}\ln(x-1) - \frac{3}{4}\ln(x+2).$$

Tenglikdan hosila hisoblaymiz

$$\frac{y'}{y} = \frac{2}{x+3} + \frac{1}{3(x-1)} - \frac{3}{4(x+2)};$$

$$y' = \frac{(x+3)^2\sqrt[3]{x-1}}{\sqrt[4]{(x+2)^3}}\left(\frac{2}{x+3} + \frac{1}{3(x-1)} - \frac{3}{4(x+2)}\right);$$

$$y' = \frac{(x+3)(19x^2+26x+3)}{12(x+2)\sqrt[3]{(x-1)^2}\sqrt[4]{(x+2)^3}}. ◄$$

$y = u^v$, bu yerda $u = u(x),\ v = v(x)$, ko‘rinishdagi funksiyaning hosilasini hisoblashda avval logarifmlash quyidagi formulaga olib keladi:

$$y' = u^v \ln u \cdot v' + vu^{v-1} \cdot u'.$$

2-misol

Ushbu $y = x^{\sin x^3}$ funksiya limitini hisoblang.

► $\ln y = \sin x^3 \cdot \ln x, \quad y' = x^{\sin x^3} \ln x \cdot 3x^2 \cos x^3 + \sin x^3 \cdot x^{\sin x^3 - 1}$. ◄

Agar y va x orasidagi bog‘lanish oshkormas ko‘rinishda, $F(x,y)=0$ tenglama bilan berilgan bo‘lsa, bunday funksiya *oshkormas funksiya* deyiladi. y' hosila $F(x,y)=0$ tenglikning ikki tarafidan, y x ning funksiyasi ekanligini e’tiborga olgan holda, hosila olib topiladi.

3-misol

Agar $x^3 + y^3 - 3xy = 0$ bo‘lsa, y' hosilani hisoblang.

► Tenglikning ikki tarafidan hosila olamiz

$$3x^2 + 3y^2 y' - 3y - 3xy' = 0,$$

so‘ ngra tenglamadan y' ni topamiz

$$y' = (y - x^2)/(y^2 - x). ◄$$

Agar y funksiyaning x argumentga bog‘liqligi parametrik ko‘rinishda, $x = x(t),\ y = y(t)$ tenglamalar bilan berilgan bo‘lsa, bunday funksiya *parametrik funksiya* deyiladi. y' yoki y'_x hosila $y'_x = \dfrac{y'_t}{x'_t}$ formula bilan hisoblanadi.

4-misol

Quyidagi $\begin{cases} x = \sqrt{1-t^2}, \\ y = tg\sqrt{1+t} \end{cases}$ tenglama bilan berilgan funksiyaning y'_x hosilasini toping.

► $y'_x = \dfrac{y'_t}{x'_t}, \quad x'(t) = \dfrac{-t}{\sqrt{1-t^2}}, \quad y'(t) = \dfrac{1}{\cos^2\sqrt{1+t}} \cdot \dfrac{1}{2\sqrt{1+t}} = \dfrac{1}{2\sqrt{1+t}\cos^2\sqrt{1+t}}.$

$$y'_x = -\frac{\sqrt{1-t^2}}{2t\sqrt{1+t}\cos^2\sqrt{1+t}} = -\frac{\sqrt{1+t}}{2t\cos^2\sqrt{1+t}}. ◄$$

Auditoriya topshiriqlari

1. Quyidagi hosilalarni logarifmlab differerensiallash qoidasi asosida hisoblang:

$a)\ \ y=\dfrac{(x-2)^2}{(x+1)^3\sqrt{x+2}};$ $\qquad d)\ \ y=\dfrac{(x-2)^2\sqrt[3]{x+1}}{\sqrt{x-3}};$

$b)\ y=(x^3+2)^{\cos x};$ $\qquad e)\ y=(\ln x)^{tg2x}.$

2. Oshkormas ko‘rinishda berilgan funksiyaning y'_x hosilasini toping.

$a)\ x^4+y^4=x^2y^2;$ $\qquad d)\ x^y=y^x;$

$b)\ y=(x^2+3)y+x\cos y=0;$ $\qquad e)\ y=x+arctgy.$

3. Parametrik ko‘rinishda berilgan funksiyaning y'_x hosilasini toping.

$a)\ \begin{cases} x=a(t-\sin t), \\ y=a(1-\cos t), \end{cases}$ $\qquad b)\ \begin{cases} x=1-t^2, \\ y=t-t^3, \end{cases}$

$d)\ \begin{cases} x=\dfrac{1-t}{1+t^2}, \\ y=\dfrac{2t}{1+t^2}, \end{cases}$ $\qquad e)\ \begin{cases} x=\arcsin\dfrac{1}{\sqrt{1+t^2}}, \\ y=\arccos\dfrac{t}{\sqrt{1+t^2}}. \end{cases}$

Mustaqil yechish uchun testlar

1. Quyidagilardan qaysi biriga logarifmlab differensiallash qoidasi qo‘llanadi:

A) $y=tg^2(\sin^2 x);$ B) $y=\sqrt{xtgx\sin^3 x};$

D) $y=\sqrt{tgx+\sin\sqrt{x}};$ E) $y=xtg(x\sin x).$

2. Quyidagilardan qaysi biriga logarifmlab differensiallash qoidasi qo‘llanadi:

A) $y=2^{x^2}+y^{\ln 2};$ B) $y=x^y+(\cos x)^{\ln 2};$ D) $y=x^{1+\ln 2};$ E) $y=2^x x^{\ln x}.$

3. $y=2^{x^2}+y^{\ln 2}$ tenglama bilan berilgan berilgan funksiya …

A) logarifmlab differensiallanadi. B) murakkab funksiya bo‘ladi.

D) oskormas funksiya bo‘ladi. E) parametrik funksiya bo‘ladi.

4. $x^2+y^2+4x-2y-3=0$ tenglama bilan berilgan egri chiziqga $(0,3)$ nuqtasida o‘tkazilgan urinma tenglamasini yozing.

A) $y=3-x$, B) $y=2x+3$, D) $y=x+3$, E) $y=3-2x$.

5. $\begin{cases} x=a\cos^3 t, \\ y=a\sin^3 t \end{cases}$ tenglama bilan berilgan funksiya hosilasini toping.

A) $y=-ctgx$, B) $y=-tgx$, D) $y=ctg^2x$, E) $y=tg^2x$.

Shaxsiy uy topshiriqlari

Berilgan funksiyalarning hosilalarini toping.

1

1.1. $y=\sqrt[3]{x^2}+x\arcsin x-\dfrac{\ln x}{\cos x}.$

1.2. $y=\dfrac{1}{2\sqrt[3]{x^2}}+x\,arctgx-\dfrac{\sin x}{\ln x}.$

1.3. $y=\dfrac{1}{3x^3}-2^x ctgx-\dfrac{\arcsin x}{x}.$

1.4. $y=\dfrac{5}{\sqrt[5]{x^2}}+3^x arcctgx+\dfrac{\log_2 x}{tgx}.$

1.5. $y=\dfrac{1}{2x^2}+2^x\arcsin x-\dfrac{\cos x}{\lg x}.$

1.6. $y=3^x\arcsin x-\dfrac{\ln x}{\sqrt[3]{x^2}}-\dfrac{1}{x^2}.$

1.7. $y=\arcsin x\log_3 x-\dfrac{3e^x}{\sqrt[3]{x^2}}-ctgx.$

1.8. $y=\dfrac{\lg x}{\sin x}-3\sqrt[3]{x^2}-2^x arctgx.$

1.9. $y=\sin x\ln x-\dfrac{3^x}{arctgx}-\dfrac{3}{x^4}.$

1.10. $y=\dfrac{1}{3x^3}+3^x\arccos x-\dfrac{tgx}{\lg x}.$

1.11. $y=\dfrac{1}{3\sqrt[5]{x^4}}+5^x\arccos x-\dfrac{ctgx}{\log_5 x}.$

1.12. $y=tgx\log_3 x-\dfrac{5^x}{\sin x}-\dfrac{5}{2\sqrt[5]{x^2}}.$

1.13. $y=\dfrac{\lg x}{\cos x}-4\sqrt[4]{x^3}-e^x\arcsin x.$

1.14. $y=4^x\log_2 x-\dfrac{\sin x}{arctgx}-\dfrac{2}{3x^3}.$

1.15. $y=3^x\log_3 x-\dfrac{arctgx}{\cos x}-\dfrac{2}{3\sqrt[4]{x^3}}.$

1.16. $y=\dfrac{\lg x}{\cos x}-5\sqrt[5]{x^2}-\left(e^x+x\right)tgx.$

1.17. $y=arcctgx\lg x-\dfrac{e^x}{\sin x}-\dfrac{5}{4\sqrt[5]{x^3}}.$

1.18. $y=\dfrac{1}{3\sqrt{x^3}}-3^x\cos x-\dfrac{arctgx}{x}.$

1.19. $y=2e^x\sin x-\dfrac{\ln x}{arctgx}-\dfrac{3}{\sqrt[3]{x}}.$

1.20. $y=\arcsin x\log_5 x-\dfrac{3^x}{tgx}-\dfrac{1}{3\sqrt[5]{x^3}}.$

1.21. $y=\dfrac{1}{3x^5}+e^x\cos x-\dfrac{1+x^2}{arctgx}.$

1.22. $y=\dfrac{\ln x}{\sin x}-8\sqrt[4]{x^5}-e^x arcctgx.$

1.23. $y=x^2 arcctgx-\dfrac{4^x}{\cos x}-\dfrac{5}{x\sqrt[5]{x}}.$

1.24. $y=3^x ctgx+\dfrac{arctgx}{\lg x}-\dfrac{3}{2\sqrt[3]{x^2}}.$

1.25. $y=2^x\arcsin x-\dfrac{ctgx}{\ln x}-\dfrac{3}{x\sqrt[3]{x}}.$

1.26. $y=e^x\sin x-\dfrac{arcctgx}{\ln x}-\dfrac{3}{x\sqrt[3]{x}}.$

1.27. $y=e^x\cos x-\dfrac{\arcsin x}{\log_5 x}-\dfrac{3}{x^7}.$

1.28. $y=\sin x\ln x-\dfrac{arctgx}{1+x^2}+\dfrac{4}{3\sqrt[4]{x^3}}.$

1.29. $y=\dfrac{5}{x\sqrt[5]{x^2}}+4^x arctgx-\dfrac{\cos x}{\log_3 x}.$

1.30. $y=(x^2+1)arctgx-\dfrac{3^x}{\sin x}-\dfrac{4}{x\sqrt[4]{x}}.$

2

2.1. $y=\sin^3 2x\cdot tg(2x+1)^3.$

2.2. $y=2^{\sin x}\cdot tg^2(2x^3+1).$

2.3. $y=\lg\left(\sin^3 2x\right)\cdot tg\sqrt{2x+1}.$

2.4. $y=3^{-x^2}\cdot\arcsin\sqrt{2x+1}.$

2.5. $y=\ln^3\left(2x^2+1\right)\cdot arctg^2\sqrt{x}.$

2.6. $y=\log_5{}^2(3x+4)\cdot arcctg^3\sqrt{x+1}.$

2.7. $y = 3^{2x^2+1} \cdot \arcsin^2 \sqrt{\ln x}$.

2.8. $y = \arcsin^2 3x \cdot ctg\, 5x^3$.

2.9. $y = 3^{-\cos^2 x} \cdot arctg \sqrt{x^2+1}$.

2.10. $y = 5^{\cos x} \cdot \arcsin 3x^3$.

2.11. $y = \log_3(\sin^2 x) \cdot \arccos^3 \sqrt{x}$.

2.12. $y = e^{tgx} \cdot \arcsin\left(\ln^3 x\right)$.

2.13. $y = tg\,(3e^x) \cdot \arccos^3 \sqrt{\lg x}$.

2.14. $y = 3^{\sin(2x^2+1)} \cdot \ln^2 \sqrt{tgx}$.

2.15. $y = tg\,(x^2+1) \cdot \arccos^2 \sqrt{\log_3 x}$.

2.16. $y = 2^{ctgx^2} \cdot \lg^3\left(\sin^2 x\right)$.

2.17. $y = e^{1/x} \cdot arctg^2\left(3^x\right)$.

2.18. $y = 5^{tgx} \cdot \sqrt{\arcsin(1/x)}$.

2.19. $y = 5^{-1/x} \cdot \arcsin(2x+1)^3$.

2.20. $y = \log_2(\cos^3 x) \cdot arcctg^2 \sqrt{x}$.

2.21. $y = \lg\left(tg^2 3x\right) \cdot \arccos \sqrt{1-2x}$.

2.22. $y = arcctg\, 2x^3 \cdot \sin^2(e^x + x^3)$.

2.23. $y = \lg^2(3x+4) \cdot \arcsin^3 \sqrt{1-x^2}$.

2.24. $y = \cos(2e^{3x}) \cdot arctg^3 \sqrt{\log_2 x}$.

2.25. $y = \sin\left(tg^2 3x\right) \cdot \ln \sqrt{1+2x^2}$.

2.26. $y = tg\,(1/x) \cdot \arcsin^3(e^x - x)$.

2.27. $y = tg^2(2x+1) \cdot arctg^3 \sqrt{\sin x^2}$.

2.28. $y = 2^{\sin x^2} \cdot \lg^3\left(\sin(1/x)\right)$.

2.29. $y = \ln^2(3x+2) \cdot \arccos^3(1/x)$.

2.30. $y = 2^{tgx} \cdot \sqrt{\arccos(1/x)}$.

3

3.1. *a)* $y = (arctg\, 3x)^{\ln(x+3)}$, *b)* $y = \dfrac{(x+5)^3\sqrt{x-2}}{\sqrt[3]{(x+2)^2}}$.

3.2. *a)* $y = (tg\, 3x)^{\arccos(2x+3)}$, *b)* $y = \dfrac{(x-3)^2\sqrt{x+2}}{\sqrt[5]{(x+3)^2}}$.

3.3. *a)* $y = (\lg(3x+2))^{arctg(2x+1)}$, *b)* $y = \dfrac{(x+3)^2\sqrt[3]{(x-3)^2}}{\sqrt{x+4}}$.

3.4. *a)* $y = (\log_3(3x+2))^{tg\,5x}$, *b)* $y = \dfrac{(x-5)^3\sqrt[3]{(x+3)^2}}{\sqrt{x+2}}$.

3.5. *a)* $y = (\cos(3x+2))^{\sin^2 3x}$, *b)* $y = \dfrac{(x-2)^3\sqrt[5]{(x-3)^3}}{(x+5)^5}$.

3.6. *a)* $y = (tg\,(2x+5))^{\ln(3x+2)}$, *b)* $y = \dfrac{(x+3)^3(x-2)^2}{\sqrt[3]{(x-5)^2}}$.

3.7. *a)* $y = (\arccos 3x)^{\lg 5x}$, *b)* $y = \dfrac{(x-5)^3\sqrt{(x-2)^3}}{\sqrt[3]{(x+3)^2}}$.

3.8. *a)* $y = (\log_2(x+2))^{\arcsin 2x}$, *b)* $y = \dfrac{(x-5)^3(x+6)^5}{\sqrt[3]{(x-3)^2}}$.

3.9. *a)* $y = (arctg\,(2x+1))^{\cos(3x+2)}$, *b)* $y = \dfrac{(x+2)^3(x+5)^4}{\sqrt[5]{(x-5)^4}}$.

3.10. *a)* $y = (\log_3(2x+7))^{\sin(x+3)}$, *b)* $y = \dfrac{(x+3)^5}{\sqrt[3]{(x+5)^2\,(x-2)^2}}$.

3.11. $a)\quad y=(arcctg\,2x)^{\ln(5x+3)},\quad b)\; y=\dfrac{(x+3)^2\sqrt{(x-2)^3}}{\sqrt[5]{x+7}}.$

3.12. $a)\quad y=(\sin(2x+5))^{arctgx},\quad b)\; y=\dfrac{(x+3)^5(x-5)^3}{\sqrt[3]{x-7}}.$

3.13. $a)\quad y=(\cos(3x+5))^{\arcsin 2x},\quad b)\; y=\dfrac{(x+3)^2\sqrt[7]{(x-5)^3}}{(x+7)^5}.$

3.14. $a)\quad y=(\ln(3x+4))^{\sin(2x+5)},\quad b)\; y=\dfrac{(x-3)^3\sqrt[5]{(x-5)^4}}{(x+5)^7}.$

3.15. $a)\quad y=(ctg\,3x)^{\ln(5x-2)},\quad b)\; y=\dfrac{\sqrt{x+3}\cdot\sqrt[3]{(x+2)^4}}{(x-2)^5}.$

3.16. $a)\quad y=(\arcsin 3x)^{\lg(7x+3)},\quad b)\; y=\dfrac{\sqrt{x+3}\cdot(x+7)^5}{\sqrt[6]{(x-2)^5}}.$

3.17. $a)\quad y=(tg\,3x)^{1/x^2},\quad b)\; y=\dfrac{\sqrt{(x+3)^3}\cdot\sqrt[3]{(x-3)^4}}{(x+2)^5}.$

3.18. $a)\quad y=(\arccos 3x)^{\log_3(2x+1)},\quad b)\; y=\dfrac{(x-6)^3(x+3)^5}{\sqrt[3]{(x+4)^4}}.$

3.19. $a)\quad y=(tg\,3x)^{\arccos(x-2)},\quad b)\; y=\dfrac{\sqrt[3]{(x+5)^4}}{(x-4)^5(x+3)^3}.$

3.20. $a)\quad y=(\sin(5x+3))^{\cos^2 x},\quad b)\; y=\dfrac{\sqrt[5]{(x+3)^4}}{(x-2)^3(x+5)^5}.$

3.21. $a)\quad y=(\sin(3x+1))^{arctg2x},\quad b)\; y=\dfrac{(x-3)^5\sqrt[4]{(x-5)^3}}{(x+7)^6}.$

3.22. $a)\quad y=(ctg\,2x)^{1/x^3},\quad b)\; y=\dfrac{\sqrt{(x+3)^3}\cdot\sqrt[4]{(x-2)^3}}{(x+2)^7}.$

3.23. $a)\quad y=(\sin 3x)^{\log_2 x},\quad b)\; y=\dfrac{\sqrt{(x-3)^5}\cdot\sqrt[3]{(x-5)^2}}{(x+3)^3}.$

3.24. $a)\quad y=(tg\,(2x+3))^{arcctgx},\quad b)\; y=\dfrac{\sqrt[5]{(x+3)^4}\,(x-5)^3}{\sqrt[3]{x-7}}.$

3.25. $a)\quad y=(1/x)^{arctg3x},\quad b)\; y=\dfrac{(x+5)^5(x-2)^3}{\sqrt[3]{(x-7)^2}}.$

3.26. $a)\quad y=(3x+5)^{tg^2 3x},\quad b)\; y=\dfrac{(x+7)^5\sqrt{(x+2)^3}}{\sqrt[3]{(x-5)^4}}.$

3.27. $a)\quad y=(\log_3(3x+1))^{\arccos 2x},\quad b)\; y=\dfrac{(x+3)^2\cdot\sqrt[4]{(x-5)^3}}{(x+1)^5}.$

3.28. $a)\quad y=(\cos^2 x)^{\log_2(3x+2)},\quad b)\; y=\dfrac{(x-3)^3(x+5)^5}{\sqrt[3]{(x+2)^5}}.$

3.29. $a)\quad y=(1/x)^{\arccos 5x},\quad b)\ y=\dfrac{(x+7)^4(x-2)^5}{\sqrt[5]{(x-5)^3}}.$

3.30. $a)\quad y=(\sin^2 x)^{\log_5(3x+5)},\quad b)\ y=\dfrac{(x-3)^3\sqrt{(x+5)^5}}{\sqrt[6]{(x+3)^5}}.$

4

Oshkormas shaklda berilgan funksiyalarning hosilalarini hisoblang.

4.1. $2^{x+y}=2^x+2^y.$

4.2. $y^2=\sin x+x\cos y.$

4.3. $y=7x+ctgy.$

4.4. $tgy=3x+5y.$

4.5. $y^2+x^2=\sin y.$

4.6. $xy=x^2+ctgy.$

4.7. $xy-6=\cos y.$

4.8. $e^y=4x-7y.$

4.9. $y^2x^2+x=5y.$

4.10. $y^2=(x-y)/(x+y).$

4.11. $xy=ctgy.$

4.12. $\sin y=xy^2+5.$

4.13. $\sin(xy)+\cos(xy)=tg(x+y).$

4.14. $y=x+arctgy.$

4.15. $x-y=\arcsin x-\arcsin y.$

4.16. $x\sin y-\cos y+\cos 2y=0.$

4.17. $x^3+5x^2y+4xy^2+.y^3=0$

4.18. $y=1+xe^y.$

4.19. $y^2=(x-y)/(x+y).$

4.20. $y=e^y+4x.$

4.21. $y^2=x+\ln(y/x).$

4.22. $y\sin x-\cos(x+y)=0.$

4.23. $x^{2/3}+y^{2/3}=a^{2/3}.$

4.24. $y=\cos(x+y).$

4.25. $\cos y=5x-3y.$

4.26. $x^3+y^3=7xy^2+2x^2y.$

4.27. $3^x+3^y=3^{x+y}.$

4.28. $xy=\cos y.$

4.29. $x^4+y^4=x^2y^2.$

4.30. $x^y=y^x.$

5

Parametrik shaklda berilgan funksiyalarning hosilalarini hisoblang.

5.1. $\begin{cases} x=\ln\left(t+\sqrt{t^2+1}\right), \\ y=t\sqrt{t^2+1}. \end{cases}$

5.2. $\begin{cases} x=\sqrt{1-t^2}, \\ y=tg\sqrt{t+1}. \end{cases}$

5.3. $\begin{cases} x=\sqrt{2t-t^2}, \\ y=\arcsin(t-1). \end{cases}$

5.4. $\begin{cases} x=\arcsin(\sin t), \\ y=\arccos(\cos t). \end{cases}$

5.5. $\begin{cases} x=\sqrt{2t-t^2}, \\ y=1/\sqrt[3]{(1-t)^2}. \end{cases}$

5.6. $\begin{cases} x=\ln(ctgt), \\ y=1/\cos^2 t. \end{cases}$

5.7. $\begin{cases} x=ctg(2e^t), \\ y=\ln(tge^t). \end{cases}$

5.8. $\begin{cases} x=arcctg(e^{t/2}), \\ y=\sqrt{e^t+1}. \end{cases}$

5.9. $\begin{cases} x = (3t^2+1)/(3t^2), \\ y = \sin(t^3/3+t). \end{cases}$

5.10. $\begin{cases} x = \arcsin(1/t), \\ y = \sqrt{t^2-1} + \arccos(1/t). \end{cases}$

5.11. $\begin{cases} x = 1/\ln t, \\ y = \ln\left(\left(1+\sqrt{1-t^2}\right)/t\right). \end{cases}$

5.12. $\begin{cases} x = \arcsin\sqrt{t}, \\ y = \sqrt{1+\sqrt{t}}. \end{cases}$

5.13. $\begin{cases} x = \ln(tgt), \\ y = 1/\sin^2 t. \end{cases}$

5.14. $\begin{cases} x = \ln(1-t^2), \\ y = \arcsin\sqrt{1-t^2}. \end{cases}$

5.15. $\begin{cases} x = arctgt \\ y = \ln\left(\sqrt{1-t^2}/(t+1)\right). \end{cases}$

5.16. $\begin{cases} x = \ln((1-t)/(1+t)), \\ y = \sqrt{1-t^2}. \end{cases}$

5.17. $\begin{cases} x = \arccos(1/t), \\ y = \sqrt{t^2-1} + \arcsin(1/t). \end{cases}$

5.18. $\begin{cases} x = tg(2e^t), \\ y = \ln(ctge^t). \end{cases}$

5.19. $\begin{cases} x = (\arcsin t)^2, \\ y = t/\sqrt{1-t^2}. \end{cases}$

5.20. $\begin{cases} x = t(t\cos t - 2\sin t), \\ y = t(t\sin t + 2\cos t). \end{cases}$

5.21. $\begin{cases} x = \arcsin\left(t/\sqrt{1+t^2}\right), \\ y = \arccos(1/\sqrt{1+t^2}). \end{cases}$

5.22. $\begin{cases} x = 6t/(1+t^2), \\ y = 6t^2/(1+t^2). \end{cases}$

5.23. $\begin{cases} x = a(t\cos t + \sin t), \\ y = a(\sin t - t\cos t). \end{cases}$

5.24. $\begin{cases} x = tgt, \\ y = 1/\sin 2t. \end{cases}$

5.25. $\begin{cases} x = \ln(1+t^2), \\ y = arctg\sqrt{1+t^2}. \end{cases}$

5.26. $\begin{cases} x = (1+t^3)/(t^2-1), \\ y = t/(t^2-1). \end{cases}$

5.27. $\begin{cases} x = (2t+t^2)/(t^3+1), \\ y = (2t-t^2)/(t^3+1). \end{cases}$

5.28. $\begin{cases} x = 2tgt, \\ y = 2\sin^2 t + \sin 2t. \end{cases}$

5.29. $\begin{cases} x = a(t-\sin t), \\ y = a(1-\cos t). \end{cases}$

5.30. $\begin{cases} x = \ln(1+t^2), \\ y = t - arctgt. \end{cases}$

5.3 Funksiya differensiali. Yuqori tartibli hosilalar.Yuqori tartibli differensiallar

$y = f(x)$ *funksiyaning differensiali* deb, funksiya orrtirmasining argument orttirmasi $\Delta x = dx$ ga nisbatan chiziqli bosh qismiga aytiladi. $dy = f'(x)dx$ kabi belgilanadi.

Differensial ta'rifidan va hosila hisoblash qoidalaridan foydalanib, quyidagi formulalarni hosil qilamiz($u = u(x),\ v = v(x)$):

$$1)\ dC = 0;\quad 2)\ d(u \pm v) = du \pm dv;\quad 3)\ d(Cu) = Cdu;\quad 4)\ d(uv) = vdu + udv;$$

$$5)\ d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2};\quad 6)\ d\ f(u) = f'(u)u'dx = f'(u)du,\ 7)\ dx = \Delta x.$$

Funksiya orttirmasi uning differensialidan Δx ga nisbatan yuqori tartibli cheksiz kichik miqdorga farq qiladi. Shuning uchun, argumentning x_0 nuqtadagi cheksiz kichik orttirmasida, funksiyaning orttirmasi uning shu nuqtadagi differensialiga taqriban teng bo'ladi, ya'ni $\Delta y \approx dz,\ f(x_0 + \Delta x) - f(x_0) \approx f'(x_0)\Delta x,$ bundan

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x \tag{3.1}$$

taqribiy hisoblash formulasiga ega bo'lamiz. Bu formula yordamida funksiyaning $x = x_0 + \Delta x$ nuqtadagi qiymati taqribiy hisoblanadi. Hisoblashdagi funksiyaning nisbiy xatoligi

$$\delta = \left|\frac{f(x) - f(x_0)}{f(x_0)}\right| \cdot 100\% \tag{3.2}$$

formula bilan topiladi.

1-misol

Ushbu $tg\,44°$ ni differensial yordamida taqribiy hisoblang va nisbiy xatolikni toping.

► $f(x) = tgx,\ f'(x) = \dfrac{1}{\cos^2 x},\ x = 44°,\ x_0 = 45°,\ \Delta x = -1° = -1° \cdot \dfrac{\pi}{180°} \approx -0{,}017.$

Taqribiy hisoblash formulasi (3.1)dan foydalansak,

$$tg\,44° \approx tg\,45° - \frac{1}{\cos^2 45°} \cdot 0{,}017 = 1 - 2 \cdot 0{,}017 = 0{,}966.$$

Nisbiy xatolik, $\delta = \left|\dfrac{f(x) - f(x_0)}{f(x_0)}\right| \cdot 100\% = \dfrac{0{,}034}{1} \cdot 100\% = 3{,}4\%.$ ◄

Berilgan $y = f(x)$ funksiyaning hosilasidan olingan hosila *ikkinchi tartibli hosila*, $(n-1)-$ tartibli hosilasidan olingan hosila $n-$ *tartibli hosila* deyiladi va mos ravishda $y'' = (y')' = f''(x) = \dfrac{d^2y}{dx^2},\ y^{(n)} = (y^{(n-1)})' = f^{(n)}(x) = \dfrac{d^ny}{dx^n}$ kabi belgilanadi.

Yuqori tartibli hosila hisoblashda quyidagi formulalar o'rinli $u = u(x),\ v = v(x)$):

$$1)\ (u\pm v)^{(n)}=u^{(n)}\pm v^{(n)};\quad 2)\,(Cu)^{(n)}=Cu^{(n)};$$

$$3)\ (uv)^{(n)}=u^{(n)}\cdot v+\frac{n}{1!}u^{(n-1)}\cdot v'+\frac{n(n-1)}{2!}u^{(n-2)}\cdot v''+...+u\cdot v^{(n)}.$$

Oxirgi 3) formula *Leybnits formulasi* deb ataladi.

2-misol

Leybnits formulasi yordamida hisoblang: $y=\left(x^2+1\right)\cdot\log_2 x,\ y^{(10)}-?$

► Qulaylik uchun quyidagicha belgilashlar kiritamiz va hosilalarini hisoblaymiz:

$$u=\log_2 x,\ v=x^2+1.$$

$$u'=\frac{1}{x\ln 2},\ u''=-\frac{1}{x^2\ln 2},u'''=\frac{2!}{x^3\ln 2},u^{(4)}=-\frac{3!}{x^4\ln 2},...,u^{(9)}=\frac{8!}{x^9\ln 2},u^{(10)}=-\frac{9!}{x^{10}\ln 2}.$$

$$v'=2x,\ v''=2,\ v'''=v^{(4)}=...=v^{(10)}=0.$$

Leybnits formulasini $n=10$ uchun yozib olamiz

$$(uv)^{(10)}=u^{(10)}\cdot v+\frac{10}{1!}u^{(9)}\cdot v'+\frac{10\cdot 9}{2!}u^{(8)}\cdot v''+...+u\cdot v^{(10)}.$$

Yuqoridagi hosilalarni hisobga olsak, yig'indining birinchi uchta hadi qoladi, ya'ni

$$y^{(10)}=-\frac{9!}{x^{10}\ln 2}\left(x^2+1\right)+\frac{10\cdot 8!}{x^9\ln 2}\cdot 2x-\frac{90\cdot 7!}{x^8\ln 2},$$

$$y^{(10)}=-\frac{2\cdot 7!}{x^{10}\ln 2}(x^2+36).\ ◄$$

$F(x,y)=0$ tenglama x ga bog'liq y funksiyani aniqlasa, bu funksiyadan yuqori tartibli hosila olish uchun y va uning hosilalari x ning funksiyasi ekanini e'tiborga olgan holda, tegishli marta differensiallash kerak.

Parametrik ko'rinishda berilgan $x=x(t),\ y=y(t)$ funksiyadan ikkinchi tartibli hosila quyidagi formula bilan hisoblanadi:

$$y'_x=\frac{y'_t}{x'_t},\ y''_{xx}=\left(y'_x\right)'_t\cdot x'_t=\left(\frac{y'_t}{x'_t}\right)'_t\cdot\frac{1}{x'_t}\ \text{yoki}\ y''_{xx}=\frac{y''_{tt}x'_t-y'_tx''_{tt}}{(x'_t)^3}.$$

3-misol

Quyidagi $\begin{cases}x=\arcsin t,\\ y=\ln(1-t^2)\end{cases}$ parametrik funksiya berilgan bo'lsa $y''_{xx}-?$

► $$y'_x=\frac{y'_t}{x'_t},\ x'(t)=\frac{1}{\sqrt{1-t^2}},\ y'_t=\frac{-2t}{1-t^2},\quad y'_x=\frac{-2t}{\sqrt{1-t^2}}.$$

$$y''_{xx}=(y'_x)'_t\cdot\frac{1}{x'_t}=\frac{-2\cdot\frac{-t}{\sqrt{1-t^2}}+2t\sqrt{1-t^2}}{1-t^2}\cdot\sqrt{1-t^2}=\frac{2t+2t(1-t^2)}{1-t^2}=\frac{2t(2-t^2)}{1-t^2}.\ ◄$$

Funksiyaning differensialidan olingan differensial *ikkinchi tartibli differensial*, $(n-1)-$tartibli differensialdan olingan differensial $n-$*tartibli differensial* deyiladi va mos ravishda

$$d^2y=d(dy)=f''(x)dx^2,\ d^ny=d(d^{n-1}y)=f^{(n)}(x)dx^n$$

formulalar bilan hisoblanadi.

4-misol

Agar $y=4^{-x^2}$ bo‘lsa d^2y ni hisoblang

► $y'=-2x\cdot 4^{-x^2}\ln 4,\ y''=4^{1-x^2}x^2\ln^2 4-2\cdot 4^{-x^2}\ln 4$,

$d^2y=f''(x)dx^2,\ d^2y=(4^{1-x^2}x^2\ln^2 4-2\cdot 4^{-x^2}\ln 4)dx^2$. ◄

Auditoriya topshiriqlari

1. $y=arctg\left(x+\sqrt{1+x^2}\right)$ funksiyaning ikkinchi tartibli hosilasini hisoblang.

2. Oshkormas tenglama bilan berilgan funksiyalarning y'' hosilasini toping.

$a)\ \ln y=x+y$, $\qquad b)\ arctgy=x+y.$

3. Parametrik tenglama bilan berilgan funksiyalardan y'' hosilasini hisoblang.

$a)\ \begin{cases} x=t^4 \\ y=\ln t, \end{cases} \qquad b)\ \begin{cases} x=a(t-\sin t) \\ y=a(1-\cos t). \end{cases}$

4. $y=2^{3x+1}$ funksiyaning $n-$tartibli hosilasini toping.

5. Quyidagi funksiyalarning birinchi va ikkinchi tartibli differensiallarini toping:

$a)\ y=xe^{x^2}$, $\quad b)\ y=\sqrt{1-x^2}\arcsin x$, $\quad d)\ y=x^3\ln x$, $\quad e)\ y=\dfrac{\ln x}{x^2}$.

6. Leybnits formulasidan foydalanib ko‘rsatilgan tartibli hosilalarni toping.

$a)\ y=x^2e^x,\ y^{(5)}-?$ $\qquad b)\ y=(x^2+1)\sin x,\ y^{(20)}-?$

7. Differensial yordamida taqribiy hisoblang, absolyut va nisbiy xatolikni toping.

$a)\ y=\sqrt[3]{x^2+2x+5},\ x=0{,}97$ $\qquad b)\ y=\sqrt{\dfrac{x^2-3}{x^2+5}},\ x=2{,}037.$

Mustaqil yechish uchun testlar

1. Quyidagilardan qaysi biri $y=x^3+\ln x$ tenglama bilan berilgan funksiyaning uchinchi tartibli hosilasi bo‘ladi?

A) $y''=6-2/x^3$, B) $y''=6+1/(2x^3)$, C) $y''=6-4/x^3$, D) $y''=6+2/x^3$.

2. Quyidagilardan qaysi biri $y^2=4x$ tenglama bilan berilgan funksiyaning ikkinchi tartibli hosilasi bo‘ladi?

A) $y''=-2/y^2$, B) $y''=2/y^2$, C) $y''=-4/y^3$, D) $y''=4/y^3$.

3. Quyidagilardan qaysi biri $y=t^2+1,\ x=\ln t$ parametrik tenglamalar bilan berilgan funksiyaning ikkinchi tartibli hosilasi bo‘ladi?

A) $y''=-2t^3$, B) $y''=-4t^2$, C) $y''=-4t$, D) $y''=-2t^2$.

4. Quyidagilardan qaysi biri noto‘g‘ri ?

A) $d^2(\sin x) = \sin x dx^2$, B) $d^2(e^x) = e^x dx^2$, C) $d^2(cosx) = -\cos x dx^2$, D) $d^2(x^3) = 6xdx^2$.

5. $\sqrt{2{,}07^3 + 1}$ ni differensial yordamida taqribiy hisoblang.

A) $3{,}21$ B) $3{,}14$ C) $3{,}22$ D) $3{,}15$.

5.4 O‘rta qiymat haqidagi teoremalar. Lopital qoidasi

1-teorema *(**Roll teoremasi**).* $y = f(x)$ *funksiya* $[a;b]$ *kesmada aniqlangan va uzluksiz bo‘lsin. Agar funksiya* $(a;b)$ *intervalda differensiallanuvchi bo‘lib,* $f(a) = f(b)$ *tenglik o‘rinli bo‘lsa, u holda kamida bitta shunday bir* $c \in (a;b)$ *nuqta topiladiki,* $f'(c) = 0$ *bo‘ladi.*

2-teorema *(**Lagranj teoremasi**).* $y = f(x)$ *funksiya* $[a;b]$ *kesmada aniqlangan va uzluksiz bo‘lib,* $(a;b)$ *intervalda differensiallanuvchi bo‘lsa, u holda kamida bitta shunday bir* $c \in (a;b)$ *nuqta topiladiki,* $f(b) - f(a) = f'(c)(b-a)$ *tenglik o‘rinli bo‘ladi.*

Bu tenglikga *Lagranjning chekli orttirmalar formulasi* deyiladi.

Teoremani geometrik izohlaydigan bo‘lsak, uning har bir sharti o‘rinli bo‘lganda, $y = f(x)$ funksiya grafigida $A(a; f(a))$ va $B(b; f(b))$ nuqtalarini tutashturuvchi AB yoyiga tegishli hech bo‘lmaganda bitta nuqta topiladiki, chiziqning shu nuqtasiga o‘ tkazilgan urinma AB vatarga parallel bo‘ladi.

3-teorema*(**Koshi teoremasi**).* $y = f(x)$ *va* $y = g(x)$ *funksiyalar* $[a;b]$ *kesmada uzluksiz bo‘lib,* $(a;b)$ *intervalda differensiallanuvchi va* $g'(x) \neq 0$, $x \in (a;b)$ *bo‘lsa, u holda kamida bitta shunday bir* $c \in (a;b)$ *nuqta topiladiki,* $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}$ *tenglik o‘rinli bo‘ladi.*

Lopital qoidasi($\frac{0}{0}$ va $\frac{\infty}{\infty}$ tipidagi aniqmasliklarni ochish uchun). $f(x), g(x)$ funksiyalar $x = x_0$ nuqtaning biror atrofida uzluksiz va differensiallanuvchi bo‘lsin. Agar $x \to x_0$ da $f(x), g(x)$ funksiyalar nolga (yoki $\pm\infty$ ga) intilsa va $\lim\limits_{x\to x_0} \dfrac{f'(x)}{g'(x)}$ mavjud bo‘lsa, u holda $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)}$ ham mavjud va bu limitlar teng, ya’ni $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to x_0} \dfrac{f'(x)}{g'(x)}$.

Lopital qoidasi $x_0 = \pm\infty$ da ham o‘rinli.

1-misol

Ushbu $\lim\limits_{x\to 0} \dfrac{e^{2x} - 1}{\sin 3x}$ limitni hisoblang

►Kasrning surati ham, maxraji ham uzluksiz, differensiallanuvchi va $x \to x_0$ da nolga intiluvchi funksiyalar bo‘lgani uchun Lopital qoidasini qo‘llaymiz,

$$\lim_{x\to 0} \frac{e^{2x} - 1}{\sin 3x} = \lim_{x\to 0} \frac{2e^{2x}}{3\cos 3x} = \frac{2}{3}. \blacktriangleleft$$

Agar $\dfrac{f'(x)}{g'(x)}$ nisbat $x \to x_0$ da yana $\frac{0}{0}$ va $\frac{\infty}{\infty}$ tipidagi aniqmaslik bo‘lsa va $f'(x), g'(x)$ funksiyalar ham yuqoridagi shartlarni qanoatlantirsa, qoidani yana bir bor qo‘llab ikkinchi tartibli hosilaga o‘tish mumkin, va hakozo. Lekin hosilalar nisbatining limiti mavjud bo‘lmasa ham funksiyalar nisbatining limiti mavjud bo‘lishi mumkin.

2-misol

Ushbu $\lim\limits_{x\to\infty}\frac{x+\sin x}{x+\cos x}$ limitni hisoblang

► $\lim\limits_{x\to\infty}\frac{(x+\sin x)'}{(x+\cos x)'}=\lim\limits_{x\to\infty}\frac{1+\cos x}{1-\sin x}$, bu limit mavjud emas, chunki kasrning surati va maxraji $[0;2]$ kesmadagi ixtiyoriy sonni, kasrning o‘z i esa ixtiyoriy musbat sonni qabul qila oladi. Demak, Lopital qoidasini qo‘llab bo‘lmaydi. Lekin

$$\lim_{x\to\infty}\frac{x+\sin x}{x+\cos x}=\lim_{x\to\infty}\frac{1+\sin x/x}{1+\cos x/x}=1. \blacktriangleleft$$

$0\cdot\infty$ va $\infty-\infty$ tipidagi aniqmasliklar osonlik bilan $\frac{0}{0}$ yoki $\frac{\infty}{\infty}$ tipidagi aniqmasliklarga keltiriladi. Masalan, agar $f(x)g(x)$, $f(x)\to 0, g(x)\to\infty$ bo‘lsa, bu ko‘paytma $\frac{f(x)}{1/g(x)}$ yoki $\frac{g(x)}{1/f(x)}$ lardan biriga almashtiriladi, agar $f(x)-g(x)$, $f(x)\to\infty, g(x)\to\infty$ bo‘lsa, $f(x)-g(x)=f(x)\left(1-\frac{g(x)}{f(x)}\right)$, bu esa $0\cdot\infty$ tipidagi aniqmaslikdir.

3-misol

Ushbu $\lim\limits_{x\to\infty}x^2e^{-3x}$ limitni hisoblang

►Bu esa $0\cdot\infty$ tipidagi aniqmaslik.

$$\lim_{x\to\infty}x^2e^{-3x}=\lim_{x\to\infty}\frac{x^2}{e^{3x}}=\lim_{x\to\infty}\frac{2x}{3e^{3x}}=\lim_{x\to\infty}\frac{2}{9e^{3x}}=0. \blacktriangleleft$$

$f(x)^{g(x)}$ ko‘rinishdagi funksiya limitini hisoblashda $1^{\infty}, 0^0, \infty^0$ tipidagi aniqmasliklar mavjud. Bunday aniqmasliklarni avval logarifmlab, $0\cdot\infty$ tipiga keltiriladi: $A=\lim\limits_{x\to x_0}f(x)^{g(x)}$, $\ln A=\lim\limits_{x\to x_0}\ln f(x)^{g(x)}=\lim\limits_{x\to x_0}\left(g(x)\cdot\ln f(x)\right)$. So‘ ngra yuqoridagi kabi almashtitirish bajarilib, Lopital qoidasi qo‘llanadi.

4-misol

Ushbu $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{\sin x}$ limitni hisoblang

►Bu yerda ∞^0 tipidagi aniqmaslik.

$$A=\lim_{x\to 0}\left(\frac{1}{x}\right)^{\sin x},\ \ln A=\lim_{x\to 0}\left(\sin x\ln\frac{1}{x}\right)=-\lim_{x\to 0}\frac{\ln x}{1/\sin x}=-\lim_{x\to 0}\frac{1/x}{-\cos x/\sin^2 x}=$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{x\cos x}=\lim_{x\to 0}\frac{\sin 2x}{\cos x-x\sin x}=0,\quad A=\lim_{x\to 0}\left(\frac{1}{x}\right)^{\sin x}=1. \blacktriangleleft$$

Auditoriya topshiriqlari

1. $y = x - x^3$ funksiya uchun $[-1;0]$ va $[0;1]$ kesmalarda Roll teoremasi shartlari bajarilishini ko‘rsating va mos c nuqta qiymatlarini aniqlang.
2. $y = x^2$ parabolaninig $A(1;1)$ va $B(3;9)$ nuqtalari orasidagi yoyida yotuvchi shunday bir nuqtani topingki, bu nuqtadan o‘tkazilgan urunma AB vatarga parallel bo‘lsin.

Quyidagi limitlarni Lopital qoidasi yordamida hisoblang:

3. $\lim\limits_{x\to 2}\dfrac{x^4 - 3x^2 + x - 6}{x^2 - 4}$
4. $\lim\limits_{x\to\infty}(x(e^{1/x} - 1))$
5. $\lim\limits_{x\to\pi/3}\dfrac{1 - 2\cos x}{\sin 3x}$
6. $\lim\limits_{x\to 0}\dfrac{e^{x^2} + 3x^2 - 1}{\sin^2 x}$
7. $\lim\limits_{x\to 0}(\cos 2x)^{\frac{1}{x^2}}$

9. $\lim\limits_{x\to 1}\left(\dfrac{x}{x-1} - \dfrac{1}{\ln x}\right)$
10. $\lim\limits_{x\to+\infty}\dfrac{\pi - 2arctgx}{\ln\left(1 + \dfrac{1}{x}\right)}$
11. $\lim\limits_{x\to\frac{1}{2}}\left(\dfrac{1}{x} - 1\right)^{tg\pi x}$
11. $\lim\limits_{x\to 0}(e^x + x)^{1/x}$
12. $\lim\limits_{x\to\pi/2}(ctgx)^{2x-\pi}$

Mustaqil yechish uchun testlar

Quyidagi limitlarni hisoblang:

1. $\lim\limits_{x\to 3}\dfrac{x^4 - 6x^2 - 3x - 18}{x^3 - 27}$;

A) $2\frac{5}{9}$, B) $2\frac{4}{9}$, C) $3\frac{5}{9}$, D) $3\frac{4}{9}$.

2. $\lim\limits_{x\to 0}\dfrac{\ln(1+3x)}{e^{2x} - 1}$;

A) 2, B) $1\frac{1}{2}$, C) 1, D) $\frac{1}{6}$.

3. $\lim\limits_{x\to\pi/2}(tgx)^{2x-\pi}$;

A) 2, B) $1\frac{1}{2}$, C) 1, D) $\frac{1}{6}$.

4. $\lim\limits_{x\to\pi}(x - \pi)tg\dfrac{x}{2}$;

A) 0, B) 1, C) 2, D) ∞.

5. $\lim\limits_{x\to 3}\left(\dfrac{1}{x-3} - \dfrac{5}{x^2 - x - 6}\right)$;

A) 2, B) $1\frac{1}{3}$, C) 1, D) $\frac{1}{5}$.

Shaxsiy uy topshiriqlari

1

Quyidagi funksiyalarning $n-$ tartibli hosilasini toping.

1.1. $y=\ln(2x+1)$

1.2. $y=x\sqrt{e^x}$

1.3. $y=\dfrac{1}{2x+1}$

1.4. $y=e^{4x}$

1.5. $y=\ln(3+x^2)$

1.6. $y=\dfrac{x}{3x+1}$

1.7. $y=\log_3(x+4)$

1.8. $y=\lg(5x+1)$

1.9. $y=\sin 3x$

1.10. $y=\sqrt[3]{e^{2x+1}}$

1.11. $y=\dfrac{1+x}{1-x}$

1.12. $y=\sqrt{x-7}$

1.13. $y=\cos 2x$

1.14. $y=\dfrac{5x+1}{13(2x+3)}$

1.15. $y=\dfrac{4}{x+3}$.

1.16. $y=\dfrac{x}{x^2-1}$

1.17. $y=\dfrac{4+15x}{5x+1}$

1.18. $y=xe^{6x}$

1.19. $y=\sin^2 x$

1.20. $y=\log_5(2x-1)$

1.21. $y=xe^x$

1.22. $y=\cos^2 x$

1.23. $y=\dfrac{1}{x^2-3x+2}$

1.24. $y=\dfrac{1}{x-7}$.

1.25. $y=\dfrac{1+x}{\sqrt{x}}$

1.26. $y=x\ln x$

1.27. $y=3e^{-3x}$.

1.28. $y=\cos(3x+1)$

1.29. $y=3^x$

1.30. $y=a^{2x}$

2

Berilgan parametrik funksiyalarning ikkinchi tartibli y''_{xx} hosilasini toping.

2.1. $\begin{cases} x=\cos 2t \\ y=2\sec^2 t. \end{cases}$

2.2. $\begin{cases} x=\sqrt{1-t^2} \\ y=1/t \end{cases}$

2.3. $\begin{cases} x=e^t\cos t, \\ y=e^t\sin t. \end{cases}$

2.4. $\begin{cases} x=\sin^2 t, \\ y=1/ch^2 t. \end{cases}$

2.5. $\begin{cases} x=t+\sin t, \\ y=2-\cos t. \end{cases}$

2.6. $\begin{cases} x=1/t, \\ y=1/(1+t^2). \end{cases}$

2.7. $\begin{cases} x=\sqrt{t}, \\ y=1/\sqrt{1-t}. \end{cases}$

2.8. $\begin{cases} x=\sin t \\ y=\sec t. \end{cases}$

2.9. $\begin{cases} x = sh^2 t, \\ y = th^2 t. \end{cases}$

2.10. $\begin{cases} x = tgt, \\ y = 1/\sin 2t. \end{cases}$

2.11. $\begin{cases} x = \sqrt{t-1}, \\ y = t/\sqrt{1-t}. \end{cases}$

2.12. $\begin{cases} x = \ln t, \\ y = arctgt. \end{cases}$

2.13. $\begin{cases} x = \sqrt{t} \\ y = \sqrt[3]{t-1} \end{cases}$

2.14. $\begin{cases} x = \cos t/(1+2\cos t), \\ y = \sin t/(1+2\cos t). \end{cases}$

2.15. $\begin{cases} x = \sqrt[3]{t-1}, \\ y = \ln t. \end{cases}$

2.16. $\begin{cases} x = t + \sin t, \\ y = 2 + \cos t. \end{cases}$

2.17. $\begin{cases} x = \cos^2 t, \\ y = tg^2 t. \end{cases}$

2.18. $\begin{cases} x = \sqrt{t-3} \\ y = \ln(t-2) \end{cases}$

2.19. $\begin{cases} x = \sin t \\ y = \ln \cos t. \end{cases}$

2.20. $\begin{cases} x = \cos t \\ y = \ln \sin t. \end{cases}$

2.21. $\begin{cases} x = \cos t + t \sin t \\ y = \sin t - t \cos t. \end{cases}$

2.22. $\begin{cases} x = e^t \\ y = \arcsin t. \end{cases}$

2.23. $\begin{cases} x = t - \sin t, \\ y = 2 - \cos t. \end{cases}$

2.24. $\begin{cases} x = cht \\ y = \sqrt[3]{sh^2 t}. \end{cases}$

2.25. $\begin{cases} x = \cos t + t \sin t \\ y = \sin 2t. \end{cases}$

2.26. $\begin{cases} x = \cos t \\ y = \sin^4 (t/2). \end{cases}$

2.27. $\begin{cases} x = arctgt \\ y = t^2/2. \end{cases}$

2.28. $\begin{cases} x = 1/t^2, \\ y = 1/(t^2+1). \end{cases}$

2.29. $\begin{cases} x = \cos t - t \sin t \\ y = cost + t \sin t. \end{cases}$

2.30. $\begin{cases} x = 2(t - \sin t), \\ y = 4(2 + \cos t). \end{cases}$

3

Differensial yordamida $0,01$ aniqlikda taqribiy hisoblang va nisbiy xatolikni toping.

3.1. $a)\sqrt[3]{27,5}$; $b) arctg 1,02.$

3.2. $a)\sqrt[7]{130}$; $b) \arcsin 0,54.$

3.3. $a) 2,9/\sqrt{2,9^2+16}$; $b) \sin 92°.$

3.4. $a)\sqrt[5]{200}$; $b) arctg\sqrt{3,2}.$

3.5. $a) 4,01^{1,5}$; $b) arctg\sqrt{0,97}.$

3.6. $a)\sqrt[3]{70}$; $b) \ln tg 46°.$

3.7. $a)\sqrt[4]{16.64}$; $b) \sin 29°.$

3.8. $a)\left(0{,}98+\sqrt{5-0{,}98^2}\right)/2$; $b)\, e^{0,2}$.

3.9. $a)\, 0{,}98^{1,5}$; $b)\, arctg\sqrt{1{,}02}$.

3.10. $a)\, \sqrt[3]{26{,}19}$; $b)\, \cos 59°$.

3.11. $a)\, 3{,}02^4+3{,}02^3$; $b)\, ctg\, 29°$.

3.12. $a)\, \sqrt{\left(2{,}037^2-3\right)/\left(2{,}037^2+5\right)}$; $b) tg\, 44°$.

3.13. $a)\, \sqrt{(4-3{,}02)/(1+3{,}02)}$; $b) arctg\sqrt{3{,}1}$.

3.14. $a) 4{,}16^{-0,5}$; $b)\, \ln tg\, 47°15'$.

3.15. $a)\, 3{,}03^5$; $b)\, \arcsin 0{,}4983$.

3.16. $a)\, \sqrt[3]{65}$; $b)\, arctg\, 0{,}98$.

3.17. $a)\, \sqrt[5]{237}$; $b)\, \sin 31°$.

3.18. $a)\, 4{,}1/\sqrt{4{,}1^2+9}$; $b)\, e^{0,25}$.

3.19. $a)\, \sqrt[3]{150}$; $b) arctg\sqrt{2{,}9}$.

3.20. $a)\, 4{,}01^3+4{,}01^2$; $b)\, \ln tg\, 44°$.

3.21. $a) 1{,}05+\sqrt{3+1{,}05^2}$; $b)\, \ln ctg\, 46°$.

3.22. $a)\, \sqrt[4]{85}$; $b)\, \ln arctg\sqrt{0{,}97}$.

3.23. $a)\sqrt[3]{8{,}36}$; $b)\, \arcsin 0{,}08$.

3.24. $a)\, \sqrt[5]{1{,}03^2}$; $b)\, \sqrt[3]{0{,}01+3\cos 0{,}01}$.

3.25. $a)\sqrt{1{,}97^2+5}$; $b)\, \cos 61°$.

3.26. $a)\, 5{,}02^3+5{,}02^2$; $b)\, ctg\, 44°$.

3.27. $a)\, \sqrt{1+0{,}01+\sin 0{,}01}$; $b) arctg\sqrt[3]{1{,}02}$.

3.28. $a)\, \sqrt[3]{8{,}24}$; $b) arcctg\sqrt{3{,}1}$.

3.29. $a) 9{,}16^{-0,5}$; $b)\, \ln ctg\, 47°15'$.

3.30. $a)\, 2{,}03^6$; $b)\, \arcsin 0{,}512$.

4

Quyidagi limitlarni Lopital qoidasi yordamida hisoblang.

4.1. $a)\, \lim\limits_{x\to\pi/4}\dfrac{1/\cos^2 x-2tgx}{1+\cos 4x}$; $b)\, \lim\limits_{x\to 0}(\ln(x+e))^{1/x}$.

4.2. $a)\, \lim\limits_{x\to\infty}\dfrac{e^{1/x^2}-1}{2arctgx^2-\pi}$; $b)\, \lim\limits_{x\to 0}\sqrt[x]{\cos\sqrt{x}}$.

4.3. $a)\, \lim\limits_{x\to\pi/4}\dfrac{1+\cos 4x}{2tgx-\sec^2 x}$; $b)\, \lim\limits_{x\to\infty}\left(\dfrac{2}{\pi}arctgx\right)^x$.

4.4. $a)\lim\limits_{x\to 0}\dfrac{e^{x^2}-1-x^2}{tg^2 2x}$; $\quad b)\lim\limits_{x\to\infty}\left(\cos(4/\sqrt{x})\right)^x$

4.5. $a)\lim\limits_{x\to 0}\dfrac{\ln(1+x^2)}{\cos 3x-e^{-x}}$; $\quad b)\lim\limits_{x\to 1/2}\left(\ln 2x\ln(2x-1)\right)$

4.6. $a)\lim\limits_{x\to 0}\dfrac{1-\cos 5x}{tg^2 2x}$; $\quad b)\lim\limits_{x\to\infty}\left((\pi-2arctgx)\cdot\ln x\right)$

4.7. $a)\lim\limits_{x\to 0}\dfrac{3^x-2^x}{x\sqrt{1-x^2}}$; $\quad b)\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{e^x-1}\right)$

4.8. $a)\lim\limits_{x\to 0}\dfrac{\arcsin 4x}{5-5e^{-3x}}$; $\quad b)\lim\limits_{x\to a}\left(a^2-x^2\right)\cdot tg\dfrac{\pi x}{2a}$

4.9. $a)\lim\limits_{x\to\infty}\dfrac{\ln x}{\sqrt[3]{x}}$; $\quad b)\lim\limits_{x\to 2}\left(2-\dfrac{x}{2}\right)^{tg\frac{\pi x}{4}}$

4.10. $a)\lim\limits_{x\to 0}\dfrac{\arcsin x-2\arcsin x}{x\sqrt{1-x^2}}$; $\quad b)\lim\limits_{x\to 0}x^{1/\ln(e^x-1)}$

4.11. $a)\lim\limits_{x\to\pi}\dfrac{\sqrt[3]{\cos 2x}-1}{2\sin^2(x/4)-1}$; $\quad b)\lim\limits_{x\to 0}\arcsin x\cdot ctgx$

4.12. $a)\lim\limits_{x\to 0}\dfrac{e^{3x}-\cos 3x}{e^{2x}-\cos 2x}$; $\quad b)\lim\limits_{x\to 1}\left(\dfrac{1}{2\left(1-\sqrt{x}\right)}-\dfrac{1}{3\left(1-\sqrt[3]{x}\right)}\right)$

4.13. $a)\lim\limits_{x\to 0}\dfrac{\sin(e^{x^2}-1)}{\cos x-1}$; $\quad b)\lim\limits_{x\to 1}(1-x)^{\cos(\pi x/2)}$

4.14. $a)\lim\limits_{x\to 0}\dfrac{e^x-x^2/2-x-1}{\cos x-x^2/2-1}$; $\quad b)\lim\limits_{x\to\infty}x^{5/(1+2\ln x)}$

4.15. $a)\lim\limits_{x\to 0}\dfrac{\pi/x}{\cos(5x/2)}$; $\quad b)\lim\limits_{x\to\infty}(\ln 2x)^{1/\ln x}$

4.16. $a)\lim\limits_{x\to 1}\dfrac{\ln(1-x)+tg(\pi x/2)}{ctg\,\pi x}$; $\quad b)\lim\limits_{x\to\infty}x\sin\dfrac{5}{6x}$

4.17. $a)\lim\limits_{x\to 0}\dfrac{e^{x^2}-1-x^3}{\sin^2 2x}$; $\quad b)\lim\limits_{x\to 1}(1-x)^{\log_2 x}$

4.18. $a)\lim\limits_{x\to 0}\dfrac{x-arctgx}{x^3}$; $\quad b)\lim\limits_{x\to 1}\left(ctg\dfrac{\pi x}{4}\right)^{tg(\pi x/2)}$

4.19. $a)\lim\limits_{x\to 0}\dfrac{e^{3x}-e^{2x}}{\sin 5x}$; $\quad b)\lim\limits_{x\to 0}(ctg\,2x)^{1/\ln x}$

4.20. $a)\lim\limits_{x\to -1}\dfrac{\sqrt[3]{1+2x}+1}{\sqrt{2+x}-1}$; $\quad b)\lim\limits_{x\to 1}\dfrac{1}{\cos(\pi x/2)\ln(1-x)}$

4.21. $a)\lim\limits_{x\to\pi/6}\dfrac{1-\sin 3x}{(6x-\pi)^2}$; $\quad b)\lim\limits_{x\to\pi/4}(tg\,2x)^{4x-\pi}$

4.22. $a)\lim\limits_{x\to 0}\dfrac{e^{3\sqrt{x}}-1}{\sqrt{\sin 2x}}$; $\quad b)\lim\limits_{x\to\infty}\left(x^2e^{1/x^2}\right)$

4.23. $a)\lim\limits_{x\to 0}\dfrac{3^x-3^{\sin x}}{x^3}$; $b)\lim\limits_{x\to\infty}\left(\dfrac{2}{\pi}arctgx\right)^x$

4.24. $a)\lim\limits_{x\to\pi}\dfrac{(1-\pi)^2}{1-tg(x/4)}$; $b)\lim\limits_{x\to 1}\ln x\ln(x-1)$

4.25. $a)\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x-\sin x}$; $b)\lim\limits_{x\to 1}(tg\,\pi x)^{2arctgx-1}$

4.26. $a)\lim\limits_{x\to 0}\dfrac{e^{tgx}-e^x}{tgx-x}$; $b)\lim\limits_{x\to 0}x^{\sin 6x}$

4.27. $a)\lim\limits_{x\to 0}\dfrac{\ln\sin 3x}{\ln\sin 2x}$; $b)\lim\limits_{x\to\infty}(x-1)^{1/\ln 2(x-1)}$

4.28. $a)\lim\limits_{x\to -2}\dfrac{\sqrt[3]{2+5x}+2}{\sqrt{3+x}-1}$; $b)\lim\limits_{x\to\infty}x^3\sin(a/x)$

4.29. $a)\lim\limits_{x\to 0}\dfrac{e^{3x}-e^{2x}}{tg\,2x}$; $b)\lim\limits_{x\to\infty}\left(\dfrac{1}{x}\right)^{tgx}$

4.30. $a)\lim\limits_{x\to 0}\dfrac{\ln\cos x}{\sin 3x}$; $b)\lim\limits_{x\to\infty}x\sin\dfrac{6}{7x}$

5.5 Funksiyaning monotonligi, ekstremumni topish. Funksiyaning eng katta va eng kichik qiymati

Differensial hisobning asosiy vazifalardan biri funksiyalarni tekshiririshning umumiy usullarini ishlab chiqishdir.

Agar $y=f(x)$ funksiya argumentining $(a;b)$ oraliqdagi katta qiymatiga funksiyaning katta(kichik) qiymati mos kelsa, ya'ni $x_1<x_2$ bo'lganda $f(x_1)<f(x_2)\left(f(x_1)>f(x_2)\right)$ bo'lsa, u holda bu funksiya shu oraliqda *o'suvchi(kamayuvchi)* deyiladi.

1-teorema. *Agar $[a;b]$ kesmada hosilaga ega bo'lgan $f(x)$ funksiya shu kesmada o'suvchi(kamayuvchi) bo'lsa, uning hosilasi $[a;b]$ kesmada manfiy(musbat) bo'lmaydi, ya'ni* $f'(x)\ge 0\left(f'(x)\le 0\right)$.

2-teorema. *Agar $[a;b]$ kesmada uzluksiz va $(a;b)$ oraliqda differensiallanuvchi $f(x)$ funksiya uchun $a<x<b$ da $f'(x)>0$ bo'lsa, bu funksiya $[a;b]$ da o'suvchi(kamayuvchi) bo'ladi.*

Biror oraliqdan olingan ixtiyoriy $x_1<x_2$ uchun $f(x_1)\le f(x_2)\left(f(x_1)\ge f(x_2)\right)$ tengsizlik o'rinli bo'lsa, u holda $f(x)$ funksiya shu oraliqda *kamaymaydigan(o'smaydigan)* funksiya deyiladi.

Funksiyaning kamaymaydigan yoki o‘smaydigan oraliqlari uning *monotonlik oraliqlari* deyiladi.

Funksiyaning hosilasi nolga teng bo‘ladigan va mavjud bo‘lmaydigan nuqtalar *kritik nuqtalar* deyiladi.

1-misol

Ushbu $f(x)=2x^2-\ln x$ funksiyaning kritik nuqtasini, o‘sish va kamayish oraliqlarini toping.

►Funksiya $x>0$ qiymatlarda aniqlangan, hosilasi $f'(x)=4x-\frac{1}{x}$. Kritik nuqtasini topamiz: $f'(x)=0,\quad 4x-\frac{1}{x}=0,\quad x=\frac{1}{2}$.

Agar $f'(x)>0$, ya’ni $4x-\frac{1}{x}>0$ bo‘lsa, funksiya o‘suvchi bo‘ladi. Demak, $(1/2;\infty)$ oraliqda funksiya o‘suvchi ekan.

Agar $f'(x)<0$ yoki $4x-\frac{1}{x}<0$ bo‘lsa, funksiya kamayuvchi bo‘ladi. Demak, $(0;1/2)$ oraliqda funksiya kamayuvchi ekan.◄

Agar absolyut miqdori bo‘yicha yetarlicha kichik bo‘lgan ixtiyoriy Δx uchun $f(x_0+\Delta x)<f(x_0)$ $\left(f(x_0+\Delta x)>f(x_0)\right)$ bo‘lsa, $x=x_0$ nuqta $f(x)$ funksiyaning *maksimum(minimum) nuqtasi* deyiladi. Funksiyaning maksimum(minimum) nuqtalardagi qiymatlari esa *maksimum(minimum)qiymatlari* deyiladi.

Maksimum va minimum nuqtalari *funksiyaning ekstremumlari*, maksimal va minimal qiymatlari esa *funksiyaning ekstremal qiymatlari* deyiladi.

3-teorema. *Agar differensiallanuvchi $y=f(x)$ funksiya $x=x_0$ nuqtada maksimumga yoki minimumga ega bo‘lsa, u holda $f'(x_0)=0$ bo‘ladi yoki $f'(x_0)$ mavjud bo‘lmaydi.*

Bu ekstremumning zaruriy shartidir. Chunki funksiya biror nuqtada ekstremumga erishsa, shu nuqta har doim kritik nuqta bo‘ladi. Ammo har bir kritik nuqta ham ekstremum nuqta bo‘la olmaydi. Masalan, $y=x^3$ funksiyadagi $x=0$ nuqta.

Quyida biz funksiya ekstremumining ikkita yetarlilik sharti bilan tanishamiz

4-teorema *(funksiya ekstremumining 1-yetarlilik sharti).* *$f(x)$ funksiya $x=x_0$ kritik nuqtani o‘z ichiga olgan birorta intervalda uzluksiz va shu intervalning hamma nuqtalarida differensiallanuvchi bo‘lsin.Agar $f'(x)$ hosila $x<x_0$ da musbat, $x>x_0$ da manfiy bo‘lsa, $x=x_0$ maksimum nuqta, $x<x_0$ da manfiy, $x>x_0$ da musbat bo‘lsa, $x=x_0$ minimum nuqta bo‘ladi.*

Bu yerda ko‘rsatilgan tengsizlik $x=x_0$ nuqtaning yetarlicha kichik atrofida bajarilishi mumkin Bu teorema birinchi tartibli hosila yordamida funksiyani ekstremumga tekshirish qoidasini aniqlaydi, uni quyidagi sxemada ifodalaymiz:

Kritik nuqta x_0 dan o‘tishda $f'(x)$ ning ishorasi			Kritik nuqtaning xarakteri
$x < x_0$	$x = x_0$	$x > x_0$	
$+$	$f'(x_0) = 0$ yoki mavjud emas	$-$	Maksimum nuqtasi
$-$	$f'(x_0) = 0$ yoki mavjud emas	$+$	Minimum nuqtasi
$+$	$f'(x_0) = 0$ yoki mavjud emas	$+$	Ekstremum mavjud emas (funksiya o‘suvchi)
$-$	$f'(x_0) = 0$ yoki mavjud emas	$-$	Ekstremum mavjud emas (funksiya kamayuvchi)

2-misol

Ushbu $f(x) = x^2 - 4\ln(1+x)$ funksiyani ekstremumga tekshiring.

►Funksiya $x > -1$ da aniqlangan. Funksiya hosilasini hisoblaymiz:

$$f'(x) = 2x - \frac{4}{1+x} = \frac{2(x^2 + x - 2)}{1+x}.$$

Funksiya aniqlanish sohasiga tegishli bitta $x = 1$-kritik nuqta mavjud ekan. $-1 < x < 1$ da $f'(x) < 0$ va $x > 1$ da $f'(x) < 0$ bo‘lgani uchun $x = 1$ -minimum nuqta; $y_{\min} = 1 - 4\ln 2$.
◄

5-teorema (*funksiya ekstremumining 2-yetarlilik sharti*). *$f(x)$ funksiya ikki marta differensiallanuvchi, $f'(x_0) = 0$ va $f''(x_0) \neq 0$ bo‘lsa, $x = x_0$ da ekstremum mavjud. Agar $f''(x_0) < 0$ bo‘lsa, $x = x_0$ maksimum nuqta, $f''(x_0) > 0$ bo‘lsa $x = x_0$ minimum nuqta bo‘ladi.*

3-misol

Ushbu $f(x) = x^2 e^{-x}$ funksiyani ikkinchi tartibli hosila yordamida ekstremumga tekshiring.

►Funksiya $x \in R$ da aniqlangan. Funksiyaning birinchi va ikkinchi tartibli hosilalarini hisoblaymiz:

$$f'(x) = 2xe^{-x} - x^2 e^{-x} = (2x - x^2)e^{-x},$$

$$f''(x) = (2 - 2x)e^{-x} - (2x - x^2)e^{-x} = (2 - 4x + x^2)e^{-x}.$$

$f'(x) = 0,\ \left(2x - x^2\right)e^{-x} = 0$ tenglamadan funksiyaning $x_1 = 0$ va $x_2 = 2$ kritik nuqtalari topiladi. Bu nuqtalardagi ikkinchi tartibli hosila qiymatlarini hisoblaymiz: $f''(0) = 2 > 0$, ya’ni $x_1 = 0$ - minimum nuqta, $f''(2) = -2e^{-2} < 0$, ya’ni $x_2 = 2$ - maksimum nuqta; $y_{\min} = 0,\ y_{\max} = 4e^{-2}$.◄

$[a;\, b]$ kesmada uzluksiz funksiya bu kesmada o‘z ining eng katta va eng kichik qiymatlaruga erishadi va bu qiymatlarga yoki $(a;\, b)$ intervalda yotuvchi kritik nuqtalarda, yoki $[a;\, b]$ kesma oxirlarida erishadi.

4-misol

Ushbu $y = 2x + 3\sqrt[3]{x^2}$ funksiyaning $[-3;\, 1]$ kesmadagi eng katta va eng kichik qiymatlarini toping.

►Funksiya hosilasi $y' = 2 + \dfrac{2}{\sqrt[3]{x}} = \dfrac{2(\sqrt[3]{x}+1)}{\sqrt[3]{x}}$. U holda $x_1 = -1$ hosila $y' = 0$ bo‘ladigan, $x_2 = 0$ hosila y' mavjud bo‘lmagan, ya’ni uziladigan nuqtalar bo‘ladi. Ikkila kritik nuqtalar ham intervalga tegishli. Funksiyaning kritik nuqtalardagi va kesma oxirlaridagi qiymatlarini hisoblaymiz:

$y(-1) = 1$, $y(0) = 0$, $y(-3) = 3(\sqrt[3]{9} - 2) \approx 0{,}24$, $y(1) = 5$.

Topilganlarni taqqoslab, berilgan funksiya $[-3;\, 1]$ kesmadagi eng katta qiymatiga $x = 1$ nuqtada, eng kichik qiymatiga $x = 0$ nuqtada erishadi, degan xulosaga kelamiz. Demak, $[-3;\, 1]$ kesmada $y_{eng\,kat.} = 5$, $y_{eng\,kich.} = 0$ bo‘lar ekan.◄

5-misol

Radiusi R ga teng bo‘lgan sharga ichki chizilgan eng katta hajmli aylanma konusning balandligini aniqlang.

►Konus hajmi: $V = \frac{1}{3}\pi r^2 H$. Bu yerda konus balandligi H sharning o‘qkesimida hosil bo‘lgan aylanaga ichki chizilgan teng yonli uchburchak balandligi hamdir, konus asosining radiusi esa shu uchburchak asosining yarmiga teng. Demak,

$$r^2 = R^2 - (H - R)^2 = 2RH - H^2.$$

Bundan

$$V = \frac{1}{3}\pi\left(2RH - H^2\right)H = \frac{\pi}{3}\left(2RH^2 - H^3\right) = V(H).$$

H ning qanday qiymatida hajm eng katta bo‘lishini topish uchun hosil bo‘lgan funksiyadan hosila olib, nolga tenglaymiz:

$$V' = \frac{\pi}{3}\left(4RH - 3H^2\right) = 0.$$

$H = \frac{4}{3}R$ nuqtani topib, uni $V'' = \frac{\pi}{3}(4R - 6H)$ hosilaga qo‘ysak, $V''\left(\frac{4R}{3}\right) = -\frac{4\pi R}{3} < 0$.

Demak, $H = \frac{4}{3}R$ da konus hajmi eng katta bo‘lar ekan.◄

Auditoriya topshiriqlari

1. $y=x^4-2x^2+3$ funksiyaning kritik nuqtalari va monotonlik oraliqlarini toping.

2. $y=\sqrt[3]{x^2(x+3)}$ funksiyaning kritik nuqtalari va monotonlik oraliqlarini toping.

3. $y=x/(x^2-6x-16)$ funksiyaning kritik nuqtalari va monotonlik oraliqlarini toping.

4. $y=\sqrt[3]{(x^2-6x+5)^2}$ funksiyani ekstremumga tekshiring.

5. $y=x\ln^2 x$ funksiyani ekstremumga tekshiring.

6. $y=(2x-1)/(x-1)^2$ funksiyani ekstremumga tekshiring.

7. $y=e^{-x^2/2}$ funksiyani ekstremumga tekshiring.

8. $f(x)=\dfrac{3}{4}x^4-x^3-9x^2+7$ funksiyaning $[-2;\,2]$ kesmadagi eng katta va eng kichik qiymatlarini toping.

9. $y=x+3\sqrt[3]{x}$ funksiyaning $[-1;\,1]$ kesmadagi eng katta va eng kichik qiymatlarini toping.

10. Sig‘imi $V=16\pi\approx 50m^3$ bo‘lgan silindr shakldagi yopiq idish tayyorlash talab qilingan bo‘lsin. Tayyorlashga eng kam material sarflash uchun idishning o‘lchamlari(R-radiusi va H-balandligi) qanday bo‘lishi kerak?

Mustaqil yechish uchun testlar

1. Quyidagilardan qaysi biri $y=x^2-2\ln x$ funksiyaning kritik nuqtalari bo‘ladi?

A) $x=1$, B) $x=\pm1$, C) $x=1,\ x=0$, D) $x=\pm1,\ x=0$

2. $y=x^4-2x^3+x^2+1$ funksiyaning o‘sish oralig‘ini aniqlang.

A) $(-\infty;\,\infty)$, B) $(-\infty;\,0)\cup(1/2;\,1)$, C) $(0;\,1/2)\cup(1;\,\infty)$, D) $(-\infty;\,0)\cup(1/2;\,\infty)$

3. $y=x^4-2x^3+x^2+1$ funksiyaning kamayish oralig‘ini aniqlang.

A) $(-\infty;\,\infty)$, B) $(-\infty;\,0)\cup(1/2;\,1)$, C) $(0;\,1/2)\cup(1;\,\infty)$, D) $(-\infty;\,0)\cup(1/2;\,\infty)$

4. $y=x^3-3x^2-9x+7$ funksiyaning minimum nuqtasini aniqlang.

A) $x=-1$, B) $x=3$, C) $x=-1,\ x=3$, D) mavjud emas.

5. $y=x^3-3x^2-9x+7$ funksiyaning $[-2;\,2]$ kesmadagi eng katta qiymatini toping.

A) 5 B) 10 C) 12 D) 15

5.6 Funksiya grafigining qavariqligi va botiqligi. Assimptotalar. Funksiyani to‘la tekshirish va grafigini yasash

Agar $y = f(x)$ funksiyaning grafigi $(a;b)$ oraliqning ixtiyoriy nuqtasida o‘tkazilgan urunmadan pastda(yuqorida) yotsa u holda funksiya grafigi shu oraliqda *qavariq(botiq)* deyiladi.

Funksiya grafigining qavariq qismini botiq qismidan ajratuvchi $M(x_0, f(x_0))$ nuqta grafikning *egilish nuqtasi* deyiladi.

1-teorema. *Agar $(a;b)$ oraliqning hamma nuqtalarida $f''(x)<0$ $(f''(x)>0)$ bo‘lsa, u holda bu oraliqning $y = f(x)$ funksiya grafigi qavariq (botiq) bo‘ladi.*

2-teorema. *Agar $f''(x_0)=0$ bo‘lsa yoki $f''(x_0)$ mavjud bo‘lmasa va $x = x_0$ nuqtadan o‘ tishida $f''(x)$ ishorasini o‘zgartirsa , u holda absissasi x_0 ga teng bo‘lgan nuqta $y = f(x)$ funksiya grafigining egilish nuqtasi bo‘ladi.*

1-misol

Ushbu $y = x^4 - 12x^3 + 48x^2 - 50$ funksiya grafigining qavariq, botiqlik oraliqlari va egilish nuqtasi topilsin.

►Funksiya $x \in R$ da aniqlangan. Funksiyaning birinchi va ikkinchi tartibli hosilalarini hisoblaymiz:

$$y' = 4x^3 - 36x^2 + 96x,\ \ y'' = 12x^2 - 72x + 96 = 12(x^2 - 6x + 8),$$

$$y'' = 0,\ x^2 - 6x + 8 = (x-2)(x-4) = 0.$$

$x_1 = 2$ va $x_2 = 4$ nuqtalar yordamida funksiya aniqlanish sohasini oraliqlarga ajratib, quyidagi jadvalni tuzamiz:

x	$(-\infty; 2)$	2	$(2; 4)$	4	$(4; \infty)$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\cup$	62	$\cap$	206	$\cup$
	botiq	egilish n.	qavariq	egilish n.	botiq

Javob: $(-\infty; 2)$ va $(4; \infty)$ -funksiya grafigining botiqlik oraliqlari, $(2; 4)$ -funksiya grafigining qavariqlik oralig‘i, $M(2; 62)$ va $N(4; 206)$ -egilish nuqtalari.◄

2-misol

Ushbu $y = \sqrt[3]{(x+3)x^2}$ funksiya grafigining qavariq, botiqlik oraliqlari va egilish nuqtasi topilsin.

►Funksiya $x \in R$ da aniqlangan. Funksiyaning birinchi tartibli hosilasini hisoblaymiz:

$$y'=\left(\sqrt[3]{x+3}\cdot\sqrt[3]{x^2}\right)'=\frac{\sqrt[3]{x^2}}{3\sqrt[3]{(x+3)^2}}+\frac{2\sqrt[3]{x+3}}{3\sqrt[3]{x}}=\frac{x+2}{\sqrt[3]{x(x+3)^2}}.$$

Ikkinchi tartibli hosila hisoblashda logarifmlab differensiallash qoidasini qo‘llaymiz:

$$\ln y'=\ln\frac{x+2}{\sqrt[3]{x(x+3)^2}}=\ln(x+2)-\frac{1}{3}\ln x-\frac{2}{3}\ln(x+3),$$

$$y''=\frac{x+2}{\sqrt[3]{x(x+3)^2}}\left(\frac{1}{x+2}-\frac{1}{3x}-\frac{2}{3(x+3)}\right)=-\frac{2}{\sqrt[3]{x^4(x+3)^5}}.$$

$f''(x)$ nolga teng bo‘la olmaydi, egilish nuqtalarini hosila mavjud bo‘lmagan $x_1=-3$ va $x_2=0$ nuqtalardan qidiramiz:

x	$(-\infty;-3)$	-3	$(-3;0)$	0	$(0;\ \infty)$
$f''(x)$	$+$	mavjud emas	$-$	mavjud emas	$-$
$f(x)$	$\smile$	0	$\frown$	0	$\frown$
	botiq	egilish nuqta	qavariq	egil.nuqta emas	qavariq

Javob: $(-\infty;-3)$ oraliqda funksiya grafigi botiq, $(-3;0)$ va $(0;\ \infty)$ oraliqlarda funksiya grafigi qavariq, $M(-3;0)$- funksiya grafigining egilish nuqtasi.◄

Agar $y=f(x)$ funksiya grafigining o‘zgaruvchi nuqtasi cheksiz uzoqlashganda undan biror to‘g‘ri chiziqqacha bo‘lgan masofa nolga intilsa, bu to‘g‘ri chiziq $y=f(x)$ funksiya grafigining *assimptotasi* deyiladi.

1) Agar $\lim\limits_{x\to a}f(x)=\pm\infty$ bo‘lsa, $x=a$ to‘g‘ri chiziq funksiya grafigining *vertikal assimptotasi* deyiladi.

2) Agar $k=\lim\limits_{x\to-\infty}\frac{f(x)}{x}$ va $b=\lim\limits_{x\to-\infty}(f(x)-kx)$ yoki

$k=\lim\limits_{x\to+\infty}\frac{f(x)}{x}$ va $b=\lim\limits_{x\to+\infty}(f(x)-kx)$ limitlar mavjud bo‘lsa, u holda $y=kx+b$ funksiya grafigining *og‘ma assimptotasi* deyiladi.

Xususan, $k=0$ da $y=b$ *gorizontal assimptota* hosil bo ladi.

3-misol

Ushbu $y=\frac{x^3}{x^2-4}$ funksiya grafigining assimptotalari topilsin.

►Funksiya $x \neq 2$ da aniqlangan. $\lim\limits_{x\to\pm2}\dfrac{x^3}{x^2-4}=\pm\infty$ bo‘lgani uchun, $x=-2$ va $x=2$ to‘g‘ri chiziqlar funksiya grafigining vertikal assimptotalari bo‘ladi.

Og‘ma assimptotalarni qidiramiz:

$$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{x^2}{x^2-4}=1,\ b=\lim_{x\to\infty}(f(x)-kx)=\lim_{x\to\infty}\left(\frac{x^3}{x^2-4}-x\right)=\lim_{x\to\infty}\frac{4x}{x^2-4}=0.$$

Demak, yagona $y=x$ og‘ma assimptotasi mavjud.◄

Quyida funksiyani to‘la tekshirish va grafigini yasash uchun umumiy sxemani keltiramiz:

1. Funksiya aniqlanish sohasi va uzilish nuqtalari topiladi.
2. Juft, toqligi, davriyligi tekshiriladi.
3. Koordinata o‘qlari bilan kesishish nuqtalari topiladi.
4. Assimptotalari topiladi.
5. O‘sish, kamayish oraliqlari, ekstremumlari topiladi.
6. Qavariqlik, botiqlik oraliqlari va egilish nuqtalari topiladi.
7. Ayrim nuqtalardagi qiymatlari hisoblanadi.
8. Funksiya grafigi yasaladi.

4-misol

Quyidagi $y=\dfrac{(x+3)^2}{x-4}$ funksiyani to‘la tekshiring va grafigini yasang.

►Yuqoridagi sxema bo‘ yicha tekshiramiz:

1. Funksiya $x\in(-\infty;4)\cup(4;\infty)$ da aniqlangan.
2. Funksiya juft ham, toq ham, davriy ham emas.
3. Koordinata o‘qlari bilan kesishish nuqtalari: $(-3;\,0)$ va $(0;\,-2{,}25)$.
4. $x=4$ to‘g‘ri chiziq funksiya grafigining vertikal assimptosi, chunki

$$\lim_{x\to4-0}\frac{(x+3)^2}{x-4}=-\infty \text{ va } \lim_{x\to4+0}\frac{(x+3)^2}{x-4}=+\infty.$$

Og‘ma assimptotalarni qidiramiz:

$$k=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\frac{(x+3)^2}{x^2-4x}=\lim_{x\to\pm\infty}\frac{\left(1+\frac{3}{x}\right)^2}{1-\frac{4}{x}}=1$$

$$b=\lim_{x\to\pm\infty}(f(x)-kx)=\lim_{x\to\pm\infty}\left(\frac{(x+3)^2}{x-4}-x\right)=\lim_{x\to\pm\infty}\frac{10x+9}{x-4}=10.$$

Demak, funksiya grafigining og‘ma assimptotasi $y=x+10$.

5. O‘sish, kamayish oraliqlari, ekstremumlari topamiz :

$$y'=\frac{x^2-8x-33}{(x-4)^2},\quad x^2-8x-33=0 \text{ yoki } x_1=-3 \text{ va } x_2=11.$$

x	$(-\infty;-3)$	-3	$(-3;\ 4)$	4	$(4;\ 11)$	11	$(11;\ \infty)$
y'	$+$	0	$-$	mavjud emas	$-$	0	$+$
y	$\nearrow$	0	$\searrow$	mavjud emas	$\searrow$	28	$\nearrow$
	o‘suvchi		kamayuvchi		kamayuvchi		o‘suvchi

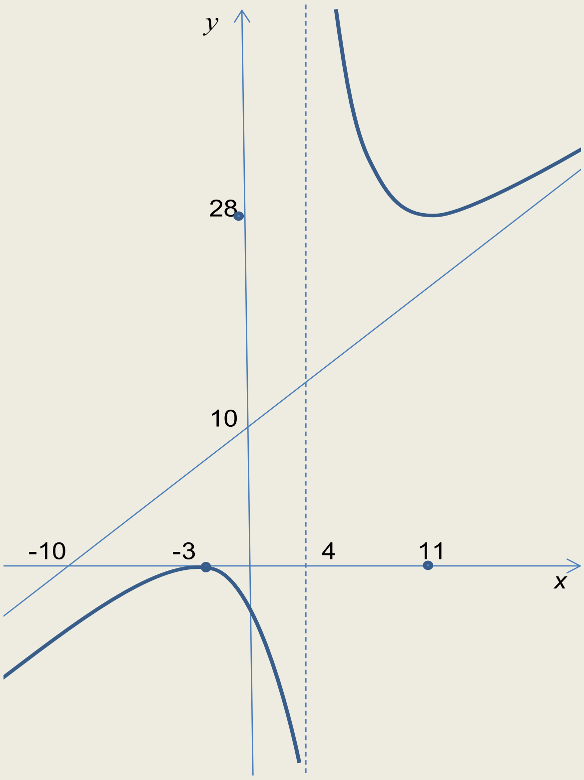
6. Qavariqlik, botiqlik oraliqlari va egilish nuqtalari topamiz:

$$y'' = \left(\frac{x^2-8x-33}{(x-4)^2}\right)' = \frac{98}{(x-4)^3}.$$

y'' nolga teng bo‘la olmaydi, y'' mavjud bo‘lmaydigan nuqta esa aniqlanish sohasiga tegishli emas. Bundan egilish nuqta mavjud emasligini aniqlaymiz. $x \in (-\infty;\ 4)$ da $y'' < 0$, funksiya grafigi qavariq, $x \in (4;\ \infty)$ da $y'' > 0$, funksiya grafigi botiq bo‘ladi.

7. Ayrim nuqtalardagi funksiya qiymatlarini hisoblaymiz: $(-10;\ -3{,}5)$, $(-4;-1/8)$, $(2;\ -12{,}5)$, $\left(10;\ 28\frac{1}{6}\right)$ va $\left(12;\ 28\frac{1}{8}\right)$.

8. Yuqoridagi tekshirishlardan foydalanib funksiya grafigini yasaymiz:



◄

Auditoriya topshiriqlari

1. $y = x^3 - 5x^2 + 3x - 5$ funksiya grafigining qavariq, botiqlik oraliqlari va egilish nuqtalari topilsin.

2. $y = \ln(1 + x^2)$ funksiya grafigining qavariq, botiqlik oraliqlari va egilish nuqtalari topilsin.

3. $y = arctgx - x$ funksiya grafigining qavariq, botiqlik oraliqlari va egilish nuqtalari topilsin.

4. $y = e^{-x^2/2}$ funksiyaning assimptotalarini toping.

5. $y = x^3/\left(2(1+x)^2\right)$ funksiyaning assimptotalarini toping.

6. $y = x\ln\left(e + \dfrac{1}{x}\right)$ funksiyaning assimptotalarini toping.

7. $y = \sqrt[3]{x^2(x+3)}$ funksiyani to‘la tekshiring va grafigini yasang.

8. $y = x^3/\left(4(2-x)^2\right)$ funksiyani to‘la tekshiring va grafigini yasang.

Mustaqil yechish uchun testlar

1. Quyidagilardan qaysi biri $y = x^2 - 2\ln x$ funksiya grafigining egilish nuqtasi bo‘ladi?

A) $(1;1)$ B) $(e; e^2 - 2)$ C) $(e^{-1}; e^{-2} + 2)$ D) mavjud emas.

2. Quyidagilardan qaysi biri $y = x^3 - 3x^2$ funksiya grafigining egilish nuqtalari bo‘ladi?

A) $(0;0)$ va $(2;-4)$ B) $(2;-4)$ C) $(1;-2)$ D) $(0;0)$

3. $y = x^3 - 3x^2$ funksiya grafigining qavariqlik oraliqlarini toping.

A) $(1;\ \infty)$, B) $(-\infty;\ 0)\cup(2;\ \infty)$, C) $(0;\ 2)$, D) $(-\infty;\ 1)$

4. $y = xe^{-x}$ funksiya grafigining botiqlik oralig‘ini toping.

A) $(1;\ \infty)$, B) $(2;\ \infty)$, C) $(-\infty;\ 2)$, D) $(-\infty;\ 1)$

5. $y = \dfrac{(x+1)^2}{x-2}$ funksiyaning og‘ma asimptotasini toping.

A) $y = -x$, B) $y = x + 4$, C) $y = x$, D) $y = x - 4$

Shaxsiy uy topshiriqlari

1

Funksiyalarning berilgan oraliqdagi eng katta va eng kichik qiymatlarini toping.

1.1. $y = 2\sin x + \cos 2x,\ [0;\ \pi/2]$

1.2. $y = x^3 e^{x+1},\ [-4;\ 0]$

1.3. $y = e^{4x - x^2},\ [1;\ 3]$

1.4. $y = (x+1)\sqrt[3]{x^2},\ [-4/5;\ 3]$

1.5. $y=4-e^{-x^2}$, $[0;\ 1]$

1.6. $y=\sqrt[3]{x-x^3}$, $[-2;\ 2]$

1.7. $y=(x-2)e^x$, $[-2;1]$

1.8. $y=x/(9-x^2)$, $[-2;\ 2]$

1.9. $y=(1+\ln x)/x$, $[1/e;\ e]$

1.10. $y=x^2+2x+2/(x-1)$, $[-1;\ 3]$

1.11. $y=(x^5-8)/x^4$, $[-3;\ 1]$

1.12. $y=(e^{2x}+1)/e^x$, $[-1;\ 2]$

1.13. $y=e^{6x-x^2}$, $[-3;\ 3]$

1.14. $y=((x+1)/x)^3$, $[1;\ 2]$

1.15. $y=(x+2)e^{1-x}$, $[-2;\ 2]$

1.16. $y=\ln(x^2-2x+4)$, $[-1;\ 3/2]$

1.17. $y=3x^4-16x^3+2$, $[-3;\ 1]$

1.18. $y=\ln(x^2-2x+2)$, $[0;\ 3]$

1.19. $y=x^4/4-6x^3+7$, $[-2;\ 4]$

1.20. $y=(3-x)e^{-x}$, $[0;\ 5]$

1.21. $y=(x^3+4)/x^2$, $[1;\ 2]$

1.22. $y=3x/(1+x^2)$, $[0;\ 5]$

1.23. $y=x^5-5x^4+5x^3+1$, $[-1;\ 2]$

1.24. $y=108x-x^4$, $[-1;\ 4]$

1.25. $y=(x-1)e^{-x}$, $[0;\ 3]$

1.26. $y=x^3/(x^2-x+1)$, $[-2;\ 2]$

1.27. $y=(2x-1)/(x-1)^2$, $[-1/2;\ 0]$

1.28. $y=\sqrt[3]{(x^2-1)^2}$, $[-3;\ 2]$

1.29. $y=2x^3+3x^2+2x+1$, $[-1;\ 5]$

1.30. $y=xe^x$, $[-2;\ 0]$.

Berilgan funksiyalarni to‘la tekshiring va grafigini yasang.

2

2.1. $y=\dfrac{4x-x^2-4}{x}$	**2.16.** $y=\dfrac{x^2+2x+2}{x-1}$
2.2. $y=\dfrac{x+1}{(x-1)^2}$	**2.17.** $y=\dfrac{x^2}{4x^2-1}$
2.3. $y=e^{\frac{1}{5+x}}$	**2.18.** $y=x+\dfrac{\ln x}{x}$
2.4. $y=\dfrac{x^2}{9-x}$	**2.19.** $y=\dfrac{x^3}{x^2-x+1}$
2.5. $y=x-\ln(1+x^2)$	**2.20.** $y=x^2-2\ln x$
2.6. $y=\dfrac{\ln x}{\sqrt{x}}$	**2.21.** $y=\dfrac{e^{2x}+1}{e^x}$
2.7. $y=x^3e^{-x^2/2}$	**2.22.** $y=(x-1)e^{3x+1}$
2.8. $y=\dfrac{4-2x}{1-x^2}$	**2.23.** $y=\dfrac{5x}{4-x^2}$
2.9. $y=\dfrac{x^3+4}{x^2}$	**2.24.** $y=\dfrac{x^3}{x^4-1}$
2.10. $y=\dfrac{(x+1)^2}{x-2}$	**2.25.** $y=\dfrac{x^4}{x^3-1}$

2.11. $y=\dfrac{4x^3+1}{x^4}$	**2.26.** $y=x+\dfrac{4}{x+2}$
2.12. $y=\dfrac{3x^2-1}{x^3}$	**2.27.** $y=x^2e^{-x}$
2.13. $y=\sqrt{x}e^{-x/2}$	**2.28.** $y=\dfrac{e\ln x}{x}$
2.14. $y=\dfrac{1+\ln x}{x}$	**2.29.** $y=\dfrac{x^3}{x^2-1}$
2.15. $y=\dfrac{3-x^2}{x+2}$	**2.30.** $y=\dfrac{(4-x)^3}{9(2-x)^2}$.

VI BOB ANIQMAS INTEGRAL. ANIQ INTEGRAL. XOSMAS INTEGRAL

6.1 Boshlang‘ich funksiya va aniqmas integral

Agar $(a;b)$ oraliqda aniqlangan $y=f(x)$ funksiya uchun $F'(x)=f(x)$ tenglik o‘rinli bo‘lsa, $F(x)$ funksiya $f(x)$ funksiyaning *boshlang‘ich funksiyasi* deyiladi.

$f(x)$ funksiyaning ikkita boshlang‘ich funksiyasi bir biridan faqat o‘z garmas songa farq qiladi.

Boshlang‘ich funksiyalar to‘plami $F(x)+C$, bu yerda C o‘z garmas son, $f(x)$ funksiyadan $(a;b)$ oraliq bo‘ yicha olingan *aniqmas integral* deyiladi va quyidagicha yoziladi:

$$\int f(x)\,dx = F(x)+C\,.$$

Quyida biz integrallashning asosiy qoidalari bilan tanishamiz:

1) $\int f'(x)\,dx=\int df(x)=f(x)+C,$

$d\int f(x)\,dx = d(F(x)+C)=f(x)dx\,;$

2) $\int (f(x)\pm g(x))dx=\int f(x)dx\pm\int g(x)dx\,;$

3) $\int kf(x)\,dx=k\int f(x)dx,\ k$ - o‘z garmas son;

4) agar $\int f(x)\,dx=F(x)+C$ bo‘lsa, u holda $\int f(ax+b)\,dx=\frac{1}{a}F(ax+b)+C$, bu yerda a va b - o‘z garmas sonlar, $a\neq 0$;

5) agar $\int f(x)\,dx=F(x)+C$ va $u=\varphi(x)$ bo‘lsa, u holda $\int f(u)\,dx=F(u)+C\,;$

Aniqmas integralning ta’rifi, integrallash qoidalari va asosiy elementar funksiyalarning hosilalari jadvalidan foydalanib aniqmas **integrallar jadvalini** tuzamiz:

1. $\int u^a\,du=\frac{u^{a+1}}{a+1}+C,\ a\neq -1.$
2. $\int \frac{du}{u}=\ln|u|+C\,.$
3. $\int a^u\,du=\frac{a^u}{\ln a}+C\,.$
4. $\int e^u\,du=e^u+C\,.$
5. $\int \sin u\,du=-\cos u+C\,.$
6. $\int \cos u\,du=\sin u+C\,.$
7. $\int \frac{du}{\cos^2 u}=tgu+C\,.$
8. $\int \frac{du}{\sin^2 u}=-ctgu+C\,.$
9. $\int \frac{du}{a^2+u^2}=\begin{cases}\frac{1}{a}arctgu+C\\ -\frac{1}{a}arcctgu+C.\end{cases}$
10. $\int \frac{du}{\sqrt{a^2-u^2}}=\begin{cases}\frac{1}{a}\arcsin u+C\\ -\frac{1}{a}\arccos u+C.\end{cases}$
11. $\int \frac{du}{u^2-a^2}=\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+C\,.$
12. $\int \frac{du}{\sqrt{u^2+k}}=\ln\left|u+\sqrt{u^2+k}\right|+C\,.$
13. $\int \frac{du}{\sin u}=\ln\left|tg\frac{u}{2}\right|+C\,.$

14. $\int \frac{du}{\cos u} = \ln\left|tg\left(\frac{u}{2}+\frac{\pi}{4}\right)\right| + C$.

15. $\int shu\,du = chu + C$.

16. $\int chu\,du = shu + C$.

17. $\int \frac{du}{ch^2 u} = thu + C$.

18. $\int \frac{du}{ch^2 u} = thu + C$.

1-misol

Quyidagi $\int\left(3x^2 + 2\sqrt{x} - \frac{5}{x^2}\right)dx$ integralni hisoblang.

$$\blacktriangleright \int\left(3x^2 + 2\sqrt{x} - \frac{5}{x^2}\right)dx = 3\int x^2dx + 2\int x^{1/2}dx - 5\int x^{-2}dx = x^3 + \frac{1}{\sqrt{x}} + \frac{10}{x^3} + C. \blacktriangleleft$$

2-misol

Quyidagi $\int \frac{1+2x^2}{x^2(1+x^2)}dx$ integralni hisoblang.

$$\blacktriangleright \int \frac{1+2x^2}{x^2(1+x^2)}dx = \int \frac{\left(1+x^2\right)+x^2}{x^2(1+x^2)}dx = \int \frac{1}{x^2}dx + \int \frac{1}{1+x^2}dx = -\frac{1}{x} + arctgx + C. \blacktriangleleft$$

3-misol

Quyidagi $\int (3x-5)^7\,dx$ integralni hisoblang.

$$\blacktriangleright \int (3x-5)^7\,dx = \frac{1}{3}\int (3x-5)^7\,d(3x-5) = \frac{1}{24}(3x-5)^8 + C. \blacktriangleleft$$

4-misol

Ushbu $\int \frac{8x - arctg\,2x}{1+4x^2}dx$ integralni hisoblang.

$\blacktriangleright$

$$\int \frac{8x - arctg\,2x}{1+4x^2}dx = \int \frac{8x}{1+4x^2}dx - \int \frac{arctg\,2x}{1+4x^2}dx = \int \frac{d\left(1+4x^2\right)}{1+4x^2} + \int arctg\,2x d(arctg\,2x) =$$

$$= \ln\left|1+4x^2\right| - \frac{1}{2}arctg^2\,2x + C. \blacktriangleleft$$

5-misol

Ushbu $\int \frac{\sin 2x}{4+\sin^2 x}dx$ integralni hisoblang.

$$\blacktriangleright \int \frac{\sin 2x}{4+\sin^2 x}dx = \int \frac{d(4+\sin^2 x)}{4+\sin^2 x} = \ln\left(4+\sin^2 x\right) + C. \blacktriangleleft$$

Yuqorida biz biror ifodani differensial ostiga kiritib, yoddan bu ifodani u deb almashtirib *bevosita integrallash* usulidan foydalandik.

Bu yerda $\varphi(x)=u$ deb almashtirish olinib, u yangi o‘zgaruvchili integral $\int f[\varphi(x)]\varphi'(x)dx=\int f(u)du$ ko‘rinishga keltirilgan bo‘ladi.

Agar $x=\varphi(u)$, $dx=\varphi'(u)du$ deb almastirsak, $\int f(x)dx=\int f(\varphi(u))\varphi'(u)du$ integralni hosil qilamiz. Bu *o‘z garuvchini almashtirish usuli* deyiladi.

6-misol

Ushbu $\int x^2\sqrt[3]{2-5x^3}dx$ integralni hisoblang.

▶ $2-5x^3=u,\ -15x^2dx=du,\ x^2dx=-\frac{1}{15}du$ almashtirishlarni bajaramiz:

$$\int x^2\sqrt[3]{2-5x^3}dx=-\frac{1}{15}\int\sqrt[3]{u}du=-\frac{1}{20}u^{\frac{4}{3}}+C=-\frac{1}{20}\sqrt[3]{\left(2-5x^3\right)^4}+C.◀$$

7-misol

Ushbu $\int x\sqrt{x-1}dx$ integralni hisoblang.

▶ $x-1=t^2,\quad x=t^2+1,\ dx=2tdt$ deb almashtiramiz.

$$\int x\sqrt{x-1}dx=\int\left(t^2+1\right)t\cdot 2tdt=2\int\left(t^4+t^2\right)dt=\frac{2}{5}t^5+\frac{2}{3}t^3+C=\frac{2}{5}(x-1)^{\frac{5}{2}}+\frac{2}{3}(x-1)^{\frac{3}{2}}+C.◀$$

8-misol

Ushbu $\int\sqrt{a^2-x^2}dx$ integralni hisoblang.

▶ Bunday keyin har qanday almashtirishlarni vertikal chiziqlar orasida berib ketamiz.

$$\int\sqrt{a^2-x^2}dx=\left|\begin{matrix}x=a\sin t\\ dx=a\cos tdt\end{matrix}\right|=\int\sqrt{a^2-a^2\sin^2 t}\,a\cos tdt=a^2\int\cos^2 tdt=a^2\int\frac{1+\cos 2t}{2}dt=$$

$$=\frac{a^2}{2}\left(t+\frac{\sin 2t}{2}\right)+C=\left|\begin{matrix}\sin t=\frac{x}{a},\ t=\arcsin\frac{x}{a}\\ \cos t=\sqrt{1-\frac{x^2}{a^2}}\end{matrix}\right|=\frac{a^2}{2}\left(\arcsin\frac{x}{a}+\frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}\right)+C=$$

$$=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C.◀$$

Bo‘laklab integrallash usuli quyidagi formulaga asolangan:

$$\int udv=uv-\int vdu,$$

bu yerda $u(x)$ va $v(x)$ - differensiallanvchi funksiyalar. Bu formula *bo'laklab integrallash formulasi* deyiladi. Bo'laklab integrallash formulasi ko'pincha quyidagi ko'rinishdagi integrallarni hisoblashda ishlatiladi:

$$1)\ \int p(x)e^{ax}dx,\ \int p(x)\sin mxdx,\ \int p(x)\cos axdx;$$

$$2)\ \int p(x)arctgxdx, \int p(x)arcctgxdx, \int p(x)\arcsin xdx,$$

$$\int p(x)\arccos xdx,\ \int p(x)\ln xdx$$

Bu integrallarni hisoblashda, 1 – turdagi integrallarda ***u*** uchun ***p(x)*** ko'phad, qolgan qismi ***dv*** uchun olinib, 2 - turdagi integrallarda ***u*** uchun mos ravishda *arctgx*, *arcctgx*, *arcsinx*, *arccosx* va *lnx*lar, qolgan qismi ***dv*** uchun olinadi.

9-misol

Ushbu $\int x\cos xdx$ integralni hisoblang.

►Bu 1-turdagi integral bo'lgani uchun quyidagicha bo'laklab integrallaymiz:

$$\int x\cos xdx = \left| \begin{matrix} u = x; & du = dx \\ dv = \cos xdx; & v = \int \cos xdx = \sin x \end{matrix} \right|$$

$$= x\sin x - \int \sin xdx = x\sin x + \cos x + C. \blacktriangleleft$$

10-misol

Ushbu $\int xarctgxdx$ integralni hisoblang.

$$\blacktriangleright \int xarctgxdx = \left| \begin{matrix} u = arctgx, & du = \dfrac{dx}{1+x^2} \\ dv = xdx, & v = \dfrac{x^2}{2} \end{matrix} \right| = \frac{x^2}{2}arctgx - \int \frac{x^2}{2(1+x^2)}dx =$$

$$= \frac{x^2}{2}arctgx - \frac{1}{2}x + \frac{1}{2}arctgx + C = \frac{x^2+1}{2}arctgx - \frac{x}{2} + C. \blacktriangleleft$$

Bo'laklab integrallash qoidasini bir necha marta qo'llash mumkin.

11-misol

Ushbu $\int x^2e^xdx$ integralni hisoblang.

► Bu yerda ikki marta bo'laklab integrallash qoidasi qo'llanadi:

$$\int x^2e^xdx = \left| \begin{matrix} u = x^2, dv = e^xdx \\ du = 2xdx, \quad v = e^x \end{matrix} \right| = x^2\cdot e^x - \int e^x\cdot 2xdx = x^2\cdot e^x - 2\int xe^xdx =$$

$$= \left| \begin{matrix} u = x, dv = e^xdx \\ du = dx, v = e^x \end{matrix} \right| = x^2\cdot e^x - 2\left[xe^x - \int e^xdx\right] = x^2\cdot e^x - 2xe^x + 2e^x + C.$$

◄

Ayrim integralni ikki marta bo‘laklab integrallansa o‘z iga qaytib keladi. Bu holda integralni noma’lum sifatida qarab, tenglama yechiladi.

12-misol

Ushbu $\int e^{2x}\sin xdx$ integralni hisoblang.

$$\blacktriangleright \int e^{2x}\sin xdx = \left| \begin{matrix} u=\sin x, & du=\cos xdx \\ dv=e^{2x}dx, & v=\frac{1}{2}e^{2x} \end{matrix} \right| = \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\int e^{2x}\cos xdx =$$

$$= \left| \begin{matrix} u=\cos x, & du=-\sin xdx \\ dv=e^{2x}dx, & v=\frac{1}{2}e^{2x} \end{matrix} \right| = \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\left(\frac{1}{2}e^{2x}\cos x + \frac{1}{2}\int e^{2x}\sin xdx\right) =$$

$$= \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}\int e^{2x}\sin xdx.$$

Oxirgi integralni chap tomonga o‘ tkazamiz

$$\frac{5}{4}\int e^{2x}\sin xdx = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x.$$

Demak,

$$\int e^{2x}\sin xdx = \frac{2}{5}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x + C. \blacktriangleleft$$

Ko‘pincha bo‘laklashni vertikal chiziqlar orasida bermay, integral ostida ham bajarish mumkin. Buning uchun biror funksiyani differensial octiga kiritiladi va bu differensialni *dv* sifatida qaraladi.

13-misol

Ushbu $\int xe^{3x}dx$ integralni hisoblang.

$$\blacktriangleright \int xe^{3x}dx = \int xd\left(\frac{1}{3}e^{3x}\right) = x\cdot\frac{1}{3}e^{3x} - \int\frac{1}{3}e^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C. \blacktriangleleft$$

Ayrim misollarda differensial funksiya dv oshkor ko‘rinishda bo‘lmasligi mumkin.

14-misol

Ushbu $\int \frac{x^2}{\left(x^2+a^2\right)^2}dx$ integralni hisoblang.

$$\blacktriangleright \int \frac{x^2}{\left(x^2+a^2\right)^2}dx = \int \frac{x\cdot x}{\left(x^2+a^2\right)^2}dx = \int xd\left(-\frac{1}{2}\cdot\frac{1}{x^2+a^2}\right) =$$

$$= -\frac{x}{2\left(x^2+a^2\right)} + \int \frac{1}{2\left(x^2+a^2\right)}dx = -\frac{x}{2\left(x^2+a^2\right)} + \frac{1}{2a}arctg\frac{x}{a} + C. \blacktriangleleft$$

Auditoriya topshiriqlari

1. Bevosita integrallab yoki o‘zgaruvchini almashtirib hisoblang

1. $\int \frac{\sqrt{\arcsin x} - 4x}{\sqrt{1-x^2}}dx.$ 5. $\int \frac{dx}{x^2\sqrt{x^2+9}}$

2. $\int x\sqrt[5]{(5x^2-3)^7}\,dx.$ 6. $\int \frac{e^x}{\sqrt{e^x+1}}dx$

3. $\int 2^x e^{2x} dx.$ 7. $\int \sqrt[3]{1+\sin x}\cos x dx$

4. $\int \frac{\sqrt{1+\ln x}}{x\ln x}dx.$ 8. $\int \frac{dx}{x^2\sqrt{x^2+9}}$

2. Bo‘laklab integrallash usuli yordamida hisoblang

1. $\int xc0s(2x-1)dx.$ 5. $\int e^{\sqrt{x}}dx$

2. $\int x\cdot 2^x dx.$ 6. $\int x^2\ln^2 x dx$

3. $\int \ln^2(x+1)dx.$ 7. $\int \sin(\ln x)dx$

4. $\int \arccos x dx.$ 8. $\int e^x \cos 2x dx$

Mustaqil yechish uchun testlar

1. Hisoblang: $\int \frac{x+3}{\sqrt{9-x^2}}dx$

A) $3\arcsin\frac{x}{3} + \sqrt{9-x^2} + C$, B) $\arcsin\frac{x}{3} - \sqrt{9-x^2} + C$,

C) $\arcsin\frac{x}{3} + \sqrt{9-x^2} + C$, D) $3\arcsin\frac{x}{3} - \sqrt{9-x^2} + C$.

2. Integralni hisoblang: $\int \frac{dx}{x\ln^2 x}$

A) $-\frac{1}{\ln x} + C$ B) $\ln(\ln x) + C$, C) $\frac{\ln^2 x}{2} + C$, D) $-\frac{1}{\ln^2 x} + C$.

3. Integralni hisoblang: $\int xe^{\frac{x}{4}}dx$

A) $4e^{\frac{x}{4}}(x-4)+C$, B) $4e^{\frac{x}{4}}(x-1)+C$, C) $e^{\frac{x}{4}}(x-4)+C$, D) $4e^{\frac{x}{4}}(x-16)+C$

4. $\int x^2 e^{3x}dx$ integralni hisoblashda necha marta bo‘laklab integrallanadi?

A) 1 marta, B) 2 marta, C) 3 marta, D) Bo‘laklab integrallanmaydi.

5. $\int 2^x e^{3x}dx$ integralni hisoblashda necha marta bo‘laklab integrallanadi?

A) 1 marta, B) 2 marta, C) 3 marta, D) Bo‘laklanmaydi.

6.2 Kasr-ratsional funksiyalarni integrallash

Quyidagi

$$R(x)=\frac{Q_m(x)}{P_n(x)}=\frac{b_0x^m+b_1x^{m-1}+\ldots+b_{m-1}x+b_m}{a_0x^n+a_1x^{n-1}+\ldots+b_{n-1}x+b_n} \qquad (2.1)$$

ko‘rinishdagi kasrga *kasr-ratsional funksiya* yoki qisqacha *ratsional funksiya* deyiladi. Bu yerda $m, n\in N$ va $a_i, b_j\in R,\ i=\overline{1,n},\ j=\overline{1,m}$.

Agar $m<n$ bo‘lsa, to‘g‘ri kasr, $m\geq n$ bo‘lsa, noto‘g‘ri kasr deyiladi.

Har qanday noto‘g‘ri kasr ko‘phadlarni bo‘lish qoidasi yordamida qandaydir ko‘phad va to‘g‘ri kasr yig‘indisi shaklida ifodalanadi:

$$R(x)=\frac{Q_m(x)}{P_n(x)}=M_{m-n}(x)+\frac{r(x)}{P_n(x)}\,. \qquad (2.2)$$

Masalan, $\dfrac{x^4+2}{x^2+3x-1}=x^2-3x+10+\dfrac{-33x+12}{x^2-3x+10}$, chunki

$$\begin{array}{rr|l}
- & x^4+2 & x^2+3x-1 \\ \cline{3-3}
 & x^4+3x^3-x^2 & x^2-3x+10 \\ \cline{1-2}
- & -3x^3+x^2+2 & \\
 & -3x^3-9x^2+3x & \\ \cline{1-2}
- & 10x^2-3x+2 & \\
 & 10x^2+30x-10 & \\ \cline{1-2}
 & -33x+12 &
\end{array}$$

$M_{m-n}(x)$ ko‘phadni integrallash oson bo‘lgani uchun, ratsional funksiyani integrallash $\dfrac{r(x)}{P_n(x)}$ to‘g‘ri kasrni integrallash masalasiga keltiriladi.

Quyidaqi to‘g‘ri kasrlar *oddiy ratsional kasrlar* deyiladi:

I. $\dfrac{A}{x-a}$,

II. $\dfrac{A}{(x-a)^k}\cdot(k\geq 2\ \text{va butun son})$

III. $\dfrac{Ax+B}{x^2+px+q}$(maxrajning diskreminanti $D=p^2-4q<0)$.

IV. $\dfrac{Ax+B}{(x^2+px+q)^k}$ $(k\geq 2$ va butun, $D<0)$.

Bu yerda A, B, a, p, q - haqiqiy sonlar.

Endi bu kasrlarning integrallarini hisoblaymiz:

1. $\int \frac{A}{x-a}dx = A\ln|x-2| + C$.

2. $\int \frac{A}{(x-a)^k}dx = A\int (x-a)^{-k}dx = -\frac{A}{(k-1)(x-a)^{k-1}} + C$.

3. $\int \frac{Ax+B}{x^2+px+q}dx$ integralda $A \neq 0$ bo‘lsa, suratida maxrajining hosilasini hosil qilib olamiz:

$$\int \frac{Ax+B}{x^2+px+q}dx = \frac{A}{2}\int \frac{(2x+p)+\left(\frac{2B}{A}-p\right)}{x^2+px+q}dx == \frac{A}{2}\int \frac{2x+p}{x^2+px+q}dx +$$

$$+\left(B-\frac{Ap}{2}\right)\int \frac{dx}{x^2+px+q} = \frac{A}{2}\ln\left(x^2+px+q\right) + \left(B-\frac{Ap}{2}\right)\int \frac{dx}{\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)}.$$

Oxirgi integralda $q-\frac{p^2}{4} = \frac{4q-p^2}{4} > 0(D<0)$ bo‘lgani uchun, jadvaldagi $\int \frac{du}{u^2+a^2}$ integralga keladi. Demak,

$$\int \frac{Ax+B}{x^2+px+q}dx = \frac{A}{2}\ln\left(x^2+px+q\right) + \frac{2B-Ap}{\sqrt{4q-p^2}}arctg\frac{2x+p}{\sqrt{4q-p^2}} + C. \qquad (2.3)$$

4. $\int \frac{Ax+B}{(x^2+px+q)^k}dx = \frac{A}{2}\int \frac{2x+p}{(x^2+px+q)^k}dx + \left(B-\frac{Ap}{2}\right)\int \frac{dx}{\left(\left(x+\frac{p}{2}\right)^2+\frac{4q-p^2}{4}\right)^k}$.

Bunda

$$\frac{A}{2}\int \frac{2x+p}{(x^2+px+q)^k}dx = -\frac{A}{2}\cdot\frac{1}{(k-1)(x^2+px+q)^{k-1}}, \qquad (2.4)$$

oxirgi integralda esa $u = x+\frac{p}{2}$, $a = \frac{\sqrt{4q-p^2}}{2}$ almashtirish bajaramiz.

$$\int \frac{du}{(u^2+a^2)^k} = \frac{1}{a^2}\int \frac{(u^2+a^2)-u^2}{(u^2+a^2)^k}du = \frac{1}{a^2}\int \frac{du}{(u^2+a^2)^{k-1}} - \frac{1}{a^2}\int \frac{u^2}{(u^2+a^2)^k}du.$$

Birinchi integral berilgan integralning tartibi bittaga kamaygan holi, ikkinchi integralni bo‘laklab integrallash mumkin. Natijada, quyidagi rekkurent formulani hosil qilamiz:

$$\int \frac{du}{(u^2+a^2)^k} = -\frac{u}{2a^2(k-1)(u^2+a^2)^{k-1}} + \frac{2k-3}{2a^2(k-1)}\int \frac{du}{(u^2+a^2)^{k-1}}. \qquad (2.5)$$

Eslatma. *Agar maxrajda* ax^2+bx+c *ko‘phad bo‘lsa, avval a qavsdan chiqariladi:* $ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$

1-misol

Ushbu $\int \frac{3x-2}{2x^2+8x+26}dx$ integralni hisoblang.

►Avval maxrajidan 2 ko‘paytuvchi qavsdan chiqaramiz, suratida maxrajining hosilasini hosil qilib olamiz.

$$\frac{1}{2}\int \frac{3x-2}{x^2+4x+13}dx = \frac{3}{4}\int \frac{2x+4-4-\frac{4}{3}}{x^2+4x+13}dx = \frac{3}{4}\int \frac{2x+4}{x^2+4x+13}dx - 4\int \frac{dx}{(x+2)^2+3^2} =$$

$$= \frac{3}{4}\ln\left(x^2+4x+13\right) - \frac{4}{3}arctg\frac{x+2}{3} + C .◄$$

2-misol

Ushbu $\int \frac{7x+3}{\left(x^2+2x+5\right)^2}dx$ integralni hisoblang.

$$► \int \frac{7x+3}{\left(x^2+2x+5\right)^2}dx = \frac{7}{2}\int \frac{2x+2-2+\frac{6}{7}}{\left(x^2+2x+5\right)^2}dx = \frac{7}{2}\int \frac{2x+2}{\left(x^2+2x+5\right)^2}dx - 4\int \frac{dx}{\left((x+1)^2+2^2\right)^2} .$$

Birinchi qo‘shiluvchi (4) formulaga ko‘ra,

$$\frac{7}{2}\int \frac{2x+2}{\left(x^2+2x+5\right)^2}dx = -\frac{7}{2}\cdot\frac{1}{x^2+2x+5} .$$

Ikkinchi integral uchun (5) rekkurent formulani qo‘llasak,

$$\int \frac{dx}{\left((x+1)^2+2^2\right)^2} = -\frac{x+1}{2\cdot 2^2(2-1)\left((x+1)^2+2^2\right)^2} + \frac{2\cdot 2-3}{2\cdot 2^2(2-1)}\int \frac{d(x+1)}{(x+1)^2+2^2} =$$

$$= -\frac{x+1}{8\left(\left(x^2+2x+5\right)\right)^2} + \frac{1}{8}\cdot\frac{1}{2}arctg\frac{x+1}{2} .$$

Demak,

$$\int \frac{7x+3}{\left(x^2+2x+5\right)^2}dx = -\frac{7}{2\left(x^2+2x+5\right)} - \frac{x+1}{8(x^2+2x+5)^2} + \frac{1}{16}arctg\frac{x+1}{2} + C .◄$$

Ma’lumki, har qanday haqiqiy koeffitsientli ko‘phad quyidagi ko‘paytma shaklida ifodalanadi:

$$P_n(x) = a_0(x-\alpha_1)^{k_1}\cdot\ldots\cdot(x-\alpha_\beta)^{k_\beta}(x^2+p_1x+q_1)^{t_1}\cdot\ldots\cdot(x^2+p_sx+q_s)^{t_s} , \qquad (2.6)$$

bu yerda $\alpha_1,\ldots,\alpha_\beta$ lar ko‘phadning $k_1,\ldots,k_\beta$ karrali haqiqiy ildizlari, $p_i^{\,2} - 4q_i < 0,\ (i=\overline{1,s})$ va $k_1+\ldots+k_\beta+2t_1+\ldots+2t_s = n$.

Teorema (*to‘g‘ri kasrni oddiy kasrlar yig‘ndisiga ajratish haqida*) *Maxraji (2.6) shaklda tasvirlangan har qanday to‘g‘ri ratsional kasrni I-IV turdagi oddiy kasrlar yig‘indisiga yoyish mumkin. Bu yoyilmada* $P_n(x)$ *ko‘phadning har bir* k_r *karrali* α_r *haqiqiy ildiziga (* $(x-\alpha_r)^{k_r}$ *ko‘paytuvcisiga)*

$$\frac{A_1}{x-\alpha_r}+\frac{A_2}{(x-\alpha_r)^2}+\frac{A_3}{(x-\alpha_r)^3}\dots+\frac{A_{k_r}}{(x-\alpha_r)^{k_r}} \qquad (2.7)$$

ko'rinishdagi k_r *ta oddiy kasrlar yig'indisi mos keladi.* $P_n(x)$ *ko'phadning har bir juft qo'shma- kompleks ildiziga* $((x^2+p_\gamma x+q_\gamma)^{t_\gamma}$ *ko'paytuvchisiga)*

$$\frac{M_1x+N_1}{x^2+p_\gamma x+q_\gamma}+\frac{M_2x+N_2}{(x^2+p_\gamma x+q_\gamma)^2}+\frac{M_3x+N_3}{(x^2+p_\gamma x+q_\gamma)^3}+\dots+\frac{M_{t_\gamma}x+N_{t_\gamma}}{(x^2+p_\gamma x+q_\gamma)^{t_\gamma}} \qquad (2.8)$$

ko'rinishdagi t_γ *ta oddiy kasrlar yig'indisi mos keladi.*

Demak, integral ostidagi $R(x)$ to‘g‘ri ratsional kasrni (2.7) va (2.8) formulalarni e’tiborga olib noma’lum koeffitsientli oddiy kasrlarga yoyiladi. So‘ng bu kasrlarga umumiy maxraj beriladi. Yoyilmadagi A, M, N koeffitsientlarning qiymatlari esa

1) *noma'lum koeffitsientlar usuli*;
2) *o'rniga qo'yish usuli*dan biri yoki ikkalasini qo‘llab aniqlanadi.

Noma’lum koeffitsientlari usulida $R(x)$ to‘g‘ri ratsional kasrning suratidagi ko‘phad hosil bo‘lgan kasrning suratidagi ko‘phadga aynan tengligidan x ning bir xil darajalari oldidagi koeffitsientlar tenglab, n ta noma’lum uchun n ta tenglamalar sistemasi hosil qilinib noma’lum koeffitsientlar topiladi.

O‘rniga qo‘yish usulida ko‘phadlar, x ning barcha qiymatlarida aynan teng bo‘lgani uchun, x ning tayin xususiy qiymatlarida tenglab noma’lum koeffitsientlar topiladi.

3-misol

Ushbu $\int\frac{x^2-3x+2}{x(x+1)^2}dx$ integralni hisoblang.

► Maxrajdagi ko‘phadning $x=0$ bir karrali haqiqiy va $x=-1$ ikki karrali ildizlari bor bo‘lgani uchun

$$\frac{x^2-3x+2}{x(x+1)^2}=\frac{A}{x}+\frac{B}{(x+1)^2}+\frac{C}{x+1}.$$

mumiy maxraj berib, suratdagi ko‘phadlarni tenglaymiz

$x^2-3x+2\equiv Ax^2+2xA+A+Bx+Cx^2+Cx$ yoki $x^2-3x+2\equiv x^2(A+C)+x(2A+B+C)+A$

.

Noma’lum koeffitsientlari usulidan foydalanamiz, x ning darajalari oldidagi koeffitsintlarni tenglaymiz:

$$\begin{aligned} x^2: &\quad A+C=1;\\ x: &\quad 2A+B+C=-3;\\ x^0: &\quad A=2. \end{aligned}$$

Bundan , $A=2,\ B=-6,\ C=-1$.

Demak,

$$\int\frac{x^2-3x+2}{x(x+1)^2}dx=\int\frac{2}{x}dx-\int\frac{6}{(x+1)^2}dx-\int\frac{1}{x+1}dx=$$

$$=2\ln|x|+\frac{6}{x+1}-\ln|x+1|+C=\ln\frac{x^2}{|x+1|}+\frac{6}{x+1}+C.◄$$

4-misol

Ushbu $\int \frac{(x^2+3)dx}{x(x-1)(x+2)}$ integralni hisoblang.

► Integral ostida to‘g‘ri ratsional kasr va u I turdagi sodda kasrlar yig‘indisiga ajraladi

$$\frac{x^2+3}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{D}{x+2},$$

bundan $x^2+3 = A(x-1)(x+2) + Bx(x+2) + Dx(x-1).$

A, B, D koeffutsientlarni topish uchun o‘rniga qo‘yish usulidan foydalanamiz:

$x=0$ bo‘lganda $3=-2A$, bundan $A=-\frac{3}{2}$;

$x=1$ bo‘lganda $4=3B$, bundan $B=\frac{4}{3}$;

$x=-2$ bo‘lganda, $7=6D$, bundan $D=\frac{7}{6}$.

Shunday qilib , quyidagini hosil qilamiz :

$$\int \frac{(x^2+3)dx}{x(x-1)(x+2)} = -\frac{3}{2}\int \frac{dx}{x} + \frac{4}{3}\int \frac{d(x-1)}{x-1} + \frac{7}{6}\int \frac{d(x+2)}{x+2} =$$

$$= -\frac{3}{2}\ln|x| + \frac{4}{3}\ln|x-1| + \frac{7}{6}\ln|x+2| + C = \ln\sqrt[6]{\frac{(x-1)^8|x+2|^7}{|x|^9}} + C. \blacktriangleleft$$

5-misol

Ushbu $\int \frac{dx}{x^3+8}$ integralni hisoblang.

► Integral ostida to‘g‘ri ratsional kasrning maxrajidagi ko‘phad ko‘paytuvchilarga ajratiladi va sodda kasrlar yig‘indisi shaklida ifodalanadi

$$\frac{1}{x^3+8} = \frac{1}{(x+2)(x^2-2x+4)} = \frac{A}{x+2} + \frac{Mx+N}{x^2-2x+4}.$$

Umumiy maxraj berib suratlari tenglanadi

$$A(x^2-2x+4) + Bx(x+2) + C(x+2) \equiv 1$$

A, M, N koeffitsientlarni topish uchun yuqoridagi usullarni birga qo‘llaymiz:

$$x=-2: \quad 12A=1$$

$$x^2: \quad A+B=0;$$

$$x^0: \quad 4A+2C=1.$$

Bundan , $A=1/12$, $B=-1/12$, $C=1/3$ va

$$\frac{1}{x^3+8} = \frac{1}{12(x+2)} - \frac{x-4}{12(x^2-2x+4)}.$$

Endi integralni hisoblaymiz:

$$\int\frac{dx}{x^3+8}=\frac{1}{12}\int\frac{dx}{x+2}-\frac{1}{12}\int\frac{x-4}{x^2-2x+4}dx=\frac{1}{12}\ln|x+2|-\frac{1}{12\cdot 2}\int\frac{(2x-2)-6}{x^2-2x+4}dx=$$

$$=\frac{1}{12}\ln|x+2|-\frac{1}{24}\int\frac{(2x-2)dx}{x^2-2x+4}+\frac{1}{4}\int\frac{d(x-1)}{(x-1)^2+\left(\sqrt{3}\right)^2}=$$

$$=\frac{1}{12}\ln|x+2|-\frac{1}{24}\ln\left|x^2-2x+4\right|+\frac{1}{4\sqrt{3}}arctg\,\frac{x-1}{\sqrt{3}}+C=$$

$$=\ln\sqrt[24]{\frac{(x+2)^2}{x^2-2x+4}}+\frac{1}{4\sqrt{3}}arctg\,\frac{x-1}{\sqrt{3}}+C\,.◄$$

Shunday qilib ratsional kasrni integrallash uchun

1) uning to‘g‘ri yoki noto‘g‘ri kasr ekanligini tekshiriladi, aks holda(ya’ni noto‘g‘ri kasr bo‘lganda) butun qismi ajratiladi, ko‘phad va to‘g‘ri ratsional kasr hosil qilinadi;

2) to‘g‘ri ratsional kasrni oddiy kasrlar yig‘indisiga ajratiladi;

3) yoyilmaning koeffitsientlari topiladi;

4) ifoda integrallanadi.

Auditoriya topshiriqlari

Integrallarni hisoblang.

1. $\int\frac{x^3}{x-2}dx.$
2. $\int\frac{x^4}{x^2+2}dx.$
3. $\int\frac{3x+5}{x^2-4x+5}dx.$
4. $\int\frac{5x+2}{x^2+2x+10}dx.$
5. $\int\frac{(x+1)^3}{x^2-x}dx$
6. $\int\frac{2x^2-5x+1}{x^3-2x^2+x}dx$
7. $\int\frac{7x-15}{x^3-2x^2+5}dx$
8. $\int\frac{3x^2+2x+1}{(x+1)^2(x^2+1)}dx$
9. $\int\frac{2x+1}{(x^2+2x+5)^3}dx$
10. $\int\frac{x+1}{x^4+4x^2+4}dx$

Mustaqil yechish uchun testlar

1. $\frac{2x^2+x+3}{x^2(x+1)^3(x^2+4)}$ ratsional kasrning oddiy kasrlarga yoyilmasi to‘g‘ri ko‘rsatilgan variantni aniqlang

A) $\frac{A}{x^2}+\frac{B}{(x+1)^3}+\frac{Cx+D}{x^2+4}$,

B) $\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{B_1}{x+1}+\frac{B_2}{(x+1)^2}+\frac{B_3}{(x+1)^3}+\frac{Cx+D}{x^2+4}$,

C) $\frac{A}{x^2}+\frac{B}{(x+1)^3}+\frac{C}{x^2+4}$,

D) $\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{B_1}{x+1}+\frac{B_2}{(x+1)^2}+\frac{B_3}{(x+1)^3}+\frac{C}{x^2+4}$.

2. $\frac{x+3}{x^3+2x^2}$ ratsional kasrning oddiy kasrlarga yoyilmasi to‘g‘ri ko‘rsatilgan variantni aniqlang.

A) $\frac{A}{x}+\frac{Cx+D}{x^2+2}$, B) $\frac{A}{x}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$, C) $\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x^2+2}$, D) $\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x+2}$.

3. Integralni hisoblang: $\int\frac{x^2+2x}{x+3}dx$.

A) $\frac{x^2}{2}-x+3\ln|x+3|+C$, B) $\frac{x^2}{2}+x+3\ln|x+3|+C$,

C) $x^2-3x+\ln|x+3|+C$, D) $\frac{x^2}{2}+2x+3\ln|x+3|+C$.

4. Integralni hisoblang: $\int\frac{x-4}{(x-2)(x-3)}dx$.

A) $\ln\frac{(x-3)^2}{|x-2|}+C$, B) $\ln\frac{(x-2)^2}{|x-3|}+C$, C) $\ln(x-3)^2|x-2|+C$, D) $\ln(x-2)^2|x-3|+C$.

5. Integralni hisoblang: $\int\frac{2x+1}{x^2+2x+5}dx$.

A) $\ln|x^2+2x+5|+C$, B) $\ln|x^2+2x+5|+\frac{1}{2}arctg\frac{x+1}{2}+C$,

C) $\frac{1}{2}\ln|x^2+2x+5|+C$, D) $\frac{1}{2}\ln|x^2+2x+5|+\frac{1}{2}arctg\frac{x+1}{2}+C$.

6.3 Trigonometrik funksiyalarni integrallash

Barcha trigonometrik funksiyalarni *sinx* va *cosx* orqali ifodalash mumkin. *sinx* va *cosx* ning ratsional funksiyasini $R(\sin x,\cos x)$ ko‘rinishda belgilaymiz.

Quyidagi

$$\int R(\sin x,\cos x)dx$$

integralni $tg\frac{x}{2}=z$ belgilash yordamida z o‘z garuvchili ratsional funksiyaning integraliga almashtirish mumkin. Integralni bunday almashtirish ratsionallashtirish deyiladi. Haqiqatdan ham, $tg\frac{x}{2}=z$ desak,

$$\cos x=\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{1-z^2}{1+z^2};\ \sin x=\frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}}$$

$$\frac{x}{2}=arctgz,\quad x=2arctgz,\quad dx=\frac{2dz}{1+z^2}.$$

Shuning uchun

$$\int R(\sin x,\cos x)dx=\int R\left(\frac{2z}{1+z^2};\frac{1-z^2}{1+z^2}\right)\cdot\frac{2dz}{1+z^2}=\int R_1(z)dz$$

hosil bo‘ladi, bunda $R_1(z)$ - z o‘z garuvchili ratsional funksiya.

Bunday almashtirish $R(\sin x,\cos x)$ ko‘rinishdagi har qanday funksiyani integrallashga imkon beradi, shuning uchun bunday almashtirish *universal trigonometrik almashtirish* deyiladi.

1-misol

Ushbu $I=\int\frac{dx}{4\sin x+3\cos x+5}$ integralni hisoblang.

► $tg\frac{x}{2}=z$ almashtirishdan foydalanamiz:

$$\sin x=\frac{2z}{1+z^2};\quad \cos x=\frac{1-z^2}{1+z^2};\quad dx=\frac{2dz}{1+z^2}.$$

$$I=\int\frac{\frac{2dz}{1+z^2}}{4\cdot\frac{2z}{1+z^2}+3\cdot\frac{1-z^2}{1+z^2}+5}=\int\frac{2dz}{(1+z^2)\cdot\frac{8z+3-3z^2+5+5z^2}{1+z^2}}=\int\frac{2dz}{2z^2+8z+8}=$$

$$=\int\frac{2dz}{2(z^2+4z+4)}=\int\frac{dz}{(z+2)^2}=-\frac{1}{z+2}+C=-\frac{1}{tg\frac{x}{2}+2}+C. \blacktriangleleft$$

Ko‘pincha, universal trigonometrik almashtirish murakkab ratsional funksiyaga olib keladi. Shuning uchun, xususiy sodda almashtirishlardan bir nechtasini keltirib o‘tamiz.

1. Agar $R(\sin x,\cos x)$ funksiya *sinx* ga nisbatan toq bo‘lsa, ya’ni

$R(-\sin x,\cos x)=-R(\sin x,\cos x)$ bo‘lsa, u holda $z=\cos x,\ dz=-\sin xdx$ almashtirish bu funksiyani ratsionallashtiradi.

2. Agar $R(\sin x,\cos x)$ funksiya *cosx* ga nisbatan toq bo‘lsa, ya’ni

$R(\sin x,-\cos x)=-R(\sin x,\cos x)$ bo‘lsa, u holda $z=\sin x,\ dz=\cos xdx$ almashtirish bu funksiyani ratsionallashtiradi.

3. Agar $R(\sin x,\cos x)$ funksiya *sinx* va *cosx* ga nisbatan juft bo‘lsa, ya’ni $R(-\sin x,-\cos x)=R(\sin x,\cos x)$ bo‘lsa. U holda $z=tgx$ almashtirish bu funksiyani ratsionallashtiradi.

Bu holda,

$$\sin^2 x = \frac{tg^2 x}{1+tg^2 x} = \frac{z^2}{1+z^2}; \ \cos^2 x = \frac{1}{1+tg^2 x} = \frac{1}{1+z^2};$$

$$x = arctgz,\ dx = \frac{dz}{1+z^2}$$

almashtirishlar o‘rinli bo‘ladi.

2-misol

Ushbu $I = \int \frac{dx}{1+\sin^2 x}$ integralni hisoblang.

► Integral belgisi ostidagi funksiya juft funksiya, shuning uchun $z = tgx$ almashtirishni bajaramiz. U holda, $x = arctgz$, $dx = \frac{dz}{1+z^2}$; $\sin^2 x = \frac{z^2}{1+z^2}$.

Natijada ,

$$I = \int \frac{dx}{1+\sin^2 x} = \int \frac{\frac{dt}{1+z^2}}{1+\frac{z^2}{1+z^2}} = \int \frac{\frac{dz}{1+z^2}}{\frac{1+z^2+z^2}{1+z^2}} = \int \frac{dz}{1+2z^2} = \frac{1}{2}\int \frac{dz}{\frac{1}{2}+z^2} = \frac{1}{2}\int \frac{dz}{\left(\sqrt{\frac{1}{2}}\right)^2 + z^2} =$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{\frac{1}{2}}} arctg\,\frac{z}{\sqrt{\frac{1}{2}}} + C = \frac{\sqrt{2}}{2} arctg\sqrt{2}Z + C = \frac{\sqrt{2}}{2} arctg\sqrt{2}tgx + C. ◄$$

3-misol

Ushbu $I = \int \frac{\sin^3 x}{2+\cos x}dx$ integralni hisoblang.

► Integral ostidagi funksiya *sinx* ga nisbatan toq funksiya . Shuning uchun $z = \cos x,\ dz = -\sin xdx$ almashtirishni bajaramiz:

$$I = \int \frac{\sin^2 x \cdot \sin xdx}{2+\cos x} = \int \frac{(1-\cos^2 x)\sin xdx}{2+\cos x} = -\int \frac{(1-z^2)dz}{2+z} = \int \frac{z^2-1}{2+z}dz =$$

$$= \int\left(z-2+\frac{3}{z+2}\right)dz = \frac{z^2}{2} - 2z + 3\ln|z+2| + C = \frac{\cos^2 x}{2} - 2\cos x + 3\ln|\cos x + 2| + C$$

.◄

4. Agar $R(\sin x, \cos x)$ funksiya *sinx* va *cosx* darajalarining ko‘paytmasi bo‘lsa, ya’ni $\int \sin^n x \cdot \cos^m xdx$ ko‘rinishdagi integralni hisoblashda, *m* va *n* ga bog‘liq holda turli almashtirishlar bajariladi:

a) agar $n > 0$ va toq bo‘lsa, u holda $\cos x = z, \sin xdx = -dz$ almashtirish bajariladi;

b) agar $m > 0$ va toq bo‘lsa, u holda $\sin x = z, \cos x dx = dz$ almashtirish bajariladi.

4-misol

Ushbu $I = \int \frac{\sin^3 x}{\cos^4 x} dx$ integralni hisoblang.

► $\cos x = z, \sin x dx = -dz$ almashtirishni bajaramiz:

$$I = \int \frac{\sin^2 x \sin x}{\cos^4 x} dx = \int \frac{(1-\cos^2 x)\sin x dx}{\cos^4 x} = -\int \frac{(1-z)^2 dz}{z^4} = -\int \frac{dz}{z^4} + \int \frac{z^2}{z^4} dz =$$

$$= \frac{1}{3z^3} - \frac{1}{z} + C = \frac{1}{3\cos^3 x} - \frac{1}{\cos x} + C. ◄$$

d) agar ikkala n va m ko‘rsatkichlar juft va nomanfiy bo‘lsa, u holda trigonometriyadan ma’lum bo‘lgan

$$\sin^2 x = \frac{1-\cos 2x}{2}; \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

darajani pasaytirish formulalaridan foydalanamiz.

5-misol

Ushbu $I = \int \sin^4 x dx$ integralni hisoblang.

► Darajani pasaytirish formulasidan foydalanamiz:

$$I = \int \sin^4 x dxd = \int (\sin^2 x)^2 dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right) dx =$$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx = \frac{1}{4}x - \frac{1}{2}\cdot\frac{\sin 2x}{2} + \frac{1}{8}x + \frac{1}{8}\cdot\frac{\sin 4x}{4} + C =$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C. ◄$$

e) agar $m + n = -2k \le 0$ (juft, nomusbat) bo‘lsa, u holda $tgx = z$ yoki $ctgx = z$ almashtirish integralni darajali funksiyalarning integrallari yig‘indisiga olib keladi. Xususan, $n < 0$, $m < 0$ va $m + n = -2k \le 0$ bo‘lsa, kasrning suratini $1 = (\sin^2 x + \cos^2 x)^s$ ifodaga almashtirish mumkin, bu yerda $s = \frac{|m+n|}{2} - 1$.

6-misol

Ushbu $I = \int \frac{\sin^2 x}{\cos^6 x} dx$ integralni hisoblang.

►Bu yerda $n = 2, m = -6, m + n = -4 < 0$, $tgx = z, x = arctgz, dx = \frac{dz}{1+z^2}$ almashtirishni bajaramiz.

$$\frac{\sin^2 x}{\cos^6 x} = \frac{\sin^2 x}{\cos^2 x}\cdot\frac{1}{\cos^4 x} = tg^2 x\left(\frac{1}{\cos^4 x}\right) = tg^2 x(1+tg^2 x)^2 = z^2(1+z^2)^2,$$

Natijada ,

$$I=\int\frac{\sin^2 x}{\cos^6 x}dx=\int z^2(1+z^2)^2\frac{dz}{1+z^2}=\int(z^2+z^4)dz=$$
$$=\frac{z^3}{3}+\frac{z^5}{5}+C=\frac{tg^3x}{3}+\frac{tg^5x}{5}+C. ◄$$

7-misol

Ushbu $I=\int\frac{dx}{\sin^3 x\cdot\cos x}$ integralni hisoblang.

►Bu yerda $n=-3,\ m=-1,\ m+n=-4<0$

$$I=\int\frac{dx}{\sin^3 x\cdot\cos x}=\int\frac{\sin^2 x+\cos^2 x}{\sin^3 x\cdot\cos x}dx=\int\frac{1}{\sin x\cdot\cos x}dx+\int\frac{\cos x}{\sin^3 x}dx=$$
$$=2\int\frac{dx}{\sin 2x}+\int\frac{d(\sin x)}{\sin^3 x}=\ln|tgx|-\frac{1}{2\sin^2 x}+C. ◄$$

f) agar darajalardan biri nolga teng, ikkinchisi manfiy toq son bo‘lsa, u holda $tg\frac{x}{2}=z$ almashtirish bajariladi.

8-misol

Ushbu $I=\int\frac{dx}{\sin^3 x}$ integralni hisoblang.

►Quyidagicha almashtirish bajaramiz:

$$tg\frac{x}{2}=z;\ dx=\frac{2dz}{1+z^2};\ \sin x=\frac{2z}{1+z^2}$$

Natijada, $I=\int\frac{dx}{\sin^3 x}=\int\frac{\frac{2dz}{1+z^2}}{\left(\frac{2z}{1+z^2}\right)^3}=\frac{1}{4}\int\frac{(1+z^2)^2}{z^3}dz=$

$$=\frac{1}{4}\int\frac{1+2z^2+z^4}{z^3}dz=\frac{1}{4}\int\left(\frac{1}{z^3}+\frac{2}{z}+z\right)dz=-\frac{1}{8z^2}+\frac{1}{2}\ln|z|+\frac{1}{4}\cdot\frac{z^2}{2}+C=$$
$$=-\frac{1}{8}ctg^2\frac{x}{2}+\frac{1}{2}\ln\left|tg\frac{x}{2}\right|+\frac{1}{8}tg^2\frac{x}{2}+C. ◄$$

5. Quyidagi ko‘rinishdagi integrallarni qarab chiqamiz:

$$\int\cos nx\cdot\cos mxdx,$$
$$\int\sin nx\cdot\cos mxdx,$$
$$\int\sin nx\cdot\sin mxdx.$$

Bunday integrallarni hisoblash uchun trigonometrik funksiyalarning ko‘paytmasini yig‘indiga almashtiruvchi formulalar qo‘llanadi:

$$\cos\alpha\cdot\cos\beta=\frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$

$$\sin\alpha\cdot\cos\beta=\frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$\sin\alpha\cdot\sin\beta=\frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$

9-misol

Ushbu $I=\int\sin 3x\cdot\cos 2x dx$ integralni hisoblang.

►Integral ostidagi ko‘paytmani yig‘indiga almashtirib integrallaymiz.

$$I=\int\sin 3x\cdot\cos 2x dx=\frac{1}{2}\int(\sin 5x+\sin x)dx=-\frac{1}{2}\cdot\frac{\cos 5x}{5}-\frac{1}{2}\cdot\cos x+C=$$

$$=-\frac{1}{10}\cdot\frac{\cos 5x}{1}-\frac{1}{2}\cdot\cos x+C.◄$$

Auditoriya topshiriqlari

Integrallarni hisoblang.

1. $\int\frac{dx}{3+5\cos x}$.
2. $\int\frac{dx}{3\sin^2 x+5\cos^2 x}$.
3. $\int\frac{dx}{8-4\sin x+7\cos x}$.
4. $\int\cos^3 x\sin^{10} x dx$.
5. $\int\sin^4 3x dx$
6. $\int\frac{dx}{\cos^4 x}dx$
7. $\int tg^3 x dx$
8. $\int\frac{\cos^2 x}{\sin^4 x}dx$
9. $\int\frac{\cos^4 x+\sin^4 x}{\cos^2 x-\sin^2 x}dx$
10. $\int\sin 3x\sin 5x dx$

Mustaqil yechish uchun testlar

1. $\int\frac{dx}{2\cos^2 x+3\sin^2 x}$ integralni hisoblashda qaysi almashtirish qo‘llanadi?

A) $tg\frac{x}{2}=t$, B) $\sin x=t$, C) $\cos x=t$, D) $tgx=t$.

2. $\int\frac{\sin^3 x}{\cos^4 x}dx$ integralni hisoblashda qaysi almashtirish qo‘llanadi?

A) $tgx=t$, B) $\sin x=t$, C) $\cos x=t$, D) To‘g‘ri javob yo‘q.

3. $\int\frac{\sin^2 x}{\cos^4 x}dx$ integralni hisoblashda qaysi almashtirish qo‘llanadi?

A) $tgx = t$, B) $tg\frac{x}{2} = t$, C) $\cos x = t$, D) $\sin x = t$.

4. Integralni hisoblang: $\int \frac{\cos xdx}{\sin^3 x}$

A) $-\frac{2}{\sin^3 x} + C$, B) $-\frac{3}{\sin^2 x} + C$, C) $\frac{3}{\sin^2 x} + C$, D) $-\frac{1}{2\sin^2 x} + C$.

5. Integralni hisoblang: $\int cos5x\sin 3xdx$.

A) $\frac{1}{16}\cos 8x - \frac{1}{4}\cos 2x + C$, B) $\frac{1}{16}\sin 8x - \frac{1}{4}\cos 2x + C$,

C) $-\frac{1}{16}\cos 8x + \frac{1}{4}\cos 2x + C$, D) $\frac{1}{16}\cos 8x - \frac{1}{4}\sin 2x + C$.

6.4 Ba'zi irratsional funksiyalarni integrallash

Har qanday irratsional funksiyalar ucnun ham elementar funksiyalar ko'rinishidagi boshlang'ich funksiyalarni aniqlab bo'lmaydi. Biz quyida ayrim almashtirishlar yordamida ratsional funksiyalar integrallariga olib kelinadigan irratsional funksiyalarning integrallarini ko'rib chiqamiz.

Quyidagi

$$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{r_1}{s_1}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{r_2}{s_2}}, \ldots \left(\frac{ax+b}{cx+d}\right)^{\frac{r_n}{s_n}}\right)dx, \tag{1}$$

bu yerda R-ratsional funksiya, a, b, c , d - o'zgarmas sonlar, r_i , s_i musbat butun sonlar, integral

$$\frac{ax+b}{cx+d} = t^m \tag{2}$$

almashtirish yordamida ratsionallashtiriladi. Bu yerda m - $\frac{r_1}{s_1}, \frac{r_2}{s_2}, \ldots \frac{r_n}{s_n}$ kasrlarning umumiy maxraji, ya'ni $m = EKUB(s_1, s_2, \ldots s_n)$.

Xususan,

$$\int R\left(x, x^{\frac{r_1}{s_1}}, x^{\frac{r_2}{s_2}}, \ldots x^{\frac{r_n}{s_n}}\right)dx$$

Integral $x = t^m$ almashtirish yordamida ratsionallashtiriladi.

1-misol

Ushbu $\int \frac{\sqrt{x}}{\sqrt[4]{x^3+4}}dx$ integralni hisoblang.

► $EKUK(2,4) = 4$ bo'lgani uchun,

$$\int \frac{\sqrt{x}}{\sqrt[4]{x^3+4}}dx = \left|\begin{matrix} x = t^4 \\ dx = 4t^3dt \end{matrix}\right| = 4\int \frac{t^5}{t^3+4}dt = 4\int\left(t^2 - \frac{4t^2}{t^3+4}\right)dt = \frac{4}{3}t^3 - \frac{16}{3}\ln\left|t^3+4\right| + C =$$

$$=\left|t=\sqrt[4]{x}\right|=\frac{4}{3}\sqrt[4]{x^3}-\frac{16}{3}\ln\left|\sqrt[4]{x^3}+4\right|+C\,.◄$$

Integral

$$\int\frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$$

ko‘rinishida berilgan bo‘lsin. Avval kasr suratida ildiz ostidagi kvadrat uchhadning differensiali hosil qilinadi($A\neq 0$), kvadrat uchhaddan to‘la kvadrat ajratiladi va quyidagi amallar bajariladi:

$$\int\frac{Ax+B}{\sqrt{ax^2+bx+c}}dx=\frac{A}{2a}\int\frac{(2ax+b)dx}{\sqrt{ax^2+bx+c}}+\left(B-\frac{Ab}{2a}\right)\int\frac{dx}{\sqrt{ax^2+bx+c}}=$$

$$=\frac{A}{a}\sqrt{ax^2+bx+c}+\left(B-\frac{Ab}{2a}\right)\int\frac{dx}{\sqrt{a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right)}}\,.$$

Agar $c\neq\frac{b^2}{4a}$, $a>0$ bo‘lsa, oxirgi integralni

$$\int\frac{du}{\sqrt{u^2+k}}=\ln\left|u+\sqrt{u^2+k}\right|+C$$

integralga keltirib hisoblash mumkin.

Agar $c>\frac{b^2}{4a}$, $a<0$ bo‘lsa,

$$\int\frac{du}{\sqrt{k^2-u^2}}=\arcsin\frac{u}{k}+C$$

integralga keltirib hisoblash mumkin.

Eslatma. *Qulaylik uchun, kvadrat uchhadni to‘la kvadratga ajratishdan avval* a *ning modulini ildizdan chiqarish kerak.*

2-misol

Ushbu $\int\frac{5x+3}{\sqrt{x^2-4x+8}}dx$ integralni hisoblang.

► $$\int\frac{5x+3}{\sqrt{x^2-4x+8}}dx=\frac{5}{2}\int\frac{2x-4}{\sqrt{x^2-4x+8}}dx+\left(3-\frac{5\cdot 4}{2}\right)\int\frac{dx}{\sqrt{x^2-4x+8}}=$$

$$=5\sqrt{x^2-4x+8}-7\int\frac{dx}{\sqrt{(x-2)^2+4}}=5\sqrt{x^2-4x+8}-7\ln\left|x-2+\sqrt{(x-2)^2+4}\right|+C$$

.◄

3-misol

Ushbu $\int\frac{3x-2}{\sqrt{10-8x-2x^2}}dx$ integralni hisoblang.

► Qulaylik uchun, avval 2 ni ildizdan chiqarib olamiz.

$$\int \frac{3x-21}{\sqrt{5-8x-2x^2}}dx = \frac{1}{\sqrt{2}}\int \frac{3x-2}{\sqrt{5-4x-x^2}}dx = \frac{\sqrt{2}}{2}I_1 .$$

Hosil bo‘lgan integralni hisoblaymiz.

$$I_1 = \int \frac{3x-2}{\sqrt{5-4x-x^2}}dx = \int \frac{-\frac{3}{2}(-4-2x)+8}{\sqrt{5-4x-x^2}}dx = -\frac{3}{2}\int \frac{(-4-2x)dx}{\sqrt{5-4x-x^2}} +$$

$$+8\int \frac{dx}{\sqrt{1-(x+2)^2}} = -3\sqrt{5-4x-x^2} + 8\arcsin(x+2) + C_1 .$$

Demak,

$$\int \frac{3x-2}{\sqrt{10-8x-2x^2}}dx = -\frac{3\sqrt{2}}{2}\sqrt{5-4x-x^2} + 4\sqrt{2}\arcsin(x+2) + C . \blacktriangleleft$$

Agar integral

$$\int \frac{Ax+B}{(x-\alpha)\sqrt{ax^2+bx+c}}dx$$

ko‘rinishda bo‘lsa, $x-\alpha = \frac{1}{t}$ almashtirish yordamida hisoblanadi.

4-misol

Ushbu $\int \frac{dx}{(x+1)\sqrt{x^2+2x+10}}$ integralni hisoblang.

$$\blacktriangleright \int \frac{dx}{(x+1)\sqrt{x^2+2x+10}} = \left| \begin{array}{l} x+1 = \frac{1}{t} \\ dx = -\frac{1}{t^2}dt \end{array} \right| = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1}{t^2}+9}} = -\int \frac{dt}{\sqrt{9t^2+1}} =$$

$$= -\frac{1}{3}\ln\left|3t+\sqrt{9t^2+1}\right| + C = -\frac{1}{3}\ln\left|\frac{3}{x+1}+\sqrt{\frac{9}{(x+1)^2}+1}\right| + C . \blacktriangleleft$$

Agar integral

$$\int R\left(x, \sqrt{ax^2+bx+c}\right)dx$$

ko‘rinishda bo‘lsa, kvadrat uchhadni to‘la kvadratga ajratib, quyidagi

1) $\int R\left(u, \sqrt{k^2-u^2}\right)du$, $u = k\sin t (u = k\cos t)$ almashtirish;

2) $\int R\left(u, \sqrt{k^2+u^2}\right)du$, $u = ktgt(u = kctgt)$ almashtirish;

3) $\int R\left(u, \sqrt{u^2-k^2}\right)du$, $u = \frac{k}{\sin t}\left(u = \frac{k}{\cos t}\right)$ almashtirish

yordamida hisoblanadigan integrallardan biriga keltirish mumkin.

5-misol

Ushbu $\int \sqrt{3+2x-x^2}\,dx$ integralni hisoblang.

$$► \int \sqrt{3+2x-x^2}\,dx = \int \sqrt{4-(x-1)^2}\,dx = \left| \begin{matrix} x-1 = 2\sin t \\ dx = 2\cos t dt \end{matrix} \right| =$$

$$= \int \sqrt{4-4\sin^2 t} \cdot 2\cos t dt = 4\int \cos^2 t dt = 2\int (1+\cos 2t) dt = 2t + \sin 2t + C =$$

$$= 2t + 2\sin t\sqrt{1-\sin^2 t} + C = 2\arcsin\frac{x-1}{2} + \frac{(x-1)\sqrt{3+2x-x^2}}{2} + C. ◄$$

6-misol

Ushbu $\int \frac{dx}{\sqrt{(x^2+4x+5)^3}}$ integralni hisoblang.

$$► \int \frac{dx}{\sqrt{(x^2+4x+5)^3}} = \int \frac{dx}{\sqrt{((x+2)^2+1)^3}} = \left| \begin{matrix} x+2 = tgt \\ dx = \dfrac{dt}{\cos^2 t} \end{matrix} \right| = \int \frac{dt}{\cos^2 t\sqrt{(tg^2 t+1)^3}} =$$

$$= \int \frac{dt}{\cos^2 t\sqrt{(tg^2 t+1)^3}} = \int \cos t dt = \sin t + C = \frac{tgt}{\sqrt{tg^2 t+1}} + C = \frac{x+2}{\sqrt{x^2+4x+5}} + C. ◄$$

Auditoriya topshiriqlari

Aniqmas integrallarni hisoblang.

1. $\int \frac{dx}{\sqrt{2x+3} + 3\sqrt[3]{2x+3}}$
2. $\int \sqrt{\frac{1-x}{1+x}} \cdot \frac{dx}{x}$
3. $\int \frac{(3x+5)dx}{\sqrt{x^2+6x+7}}$
4. $\int \frac{(2x-7)dx}{\sqrt{9-8x-x^2}}$
5. $\int \frac{(2x+3)dx}{\sqrt{2x^2-x+6}}$
6. $\int \frac{(3x+4)dx}{\sqrt{2+3x-x^2}}$
7. $\int \frac{dx}{x^2\sqrt{x^2+4}}$

8. $\int\sqrt{12-4x-x^2}$

9. $\int\sqrt{6x-x^2}dx$

10. $\int\frac{dx}{x\sqrt{x^2+x+1}}$

Mustaqil yechish uchun testlar

1. $\int\frac{dx}{\sqrt{x}+3\sqrt[3]{x}}$ integralda qanday almashtirish bajariladi?

A) $x=t^2$; B) $\sqrt[3]{x}=t$; C) $x=t^5$; D) $\sqrt[6]{x}=t$

2. Quyidagi $\int\frac{(2x-1)dx}{\sqrt{x^2-2x+5}}$ integralning yechimini toping

A) $2\sqrt{x^2-2x+5}+C$; B) $2\sqrt{x^2-2x+5}-\frac{1}{2}arctg\frac{x-1}{2}+C$;

C) $2\sqrt{x^2-2x+5}+\frac{1}{2}arctg\frac{x-1}{2}+C$; D) $2\sqrt{x^2-2x+5}-arctg\frac{x-1}{2}+C$;

3. $\int\sqrt{4-x^2}dx$ integral yechimini toping

A) $2\arcsin\frac{x}{2}+\frac{1}{2}x\sqrt{4-x^2}+C$; B) $2\arcsin\frac{x}{2}+x\sqrt{4-x^2}+C$;

C) $2\arcsin\frac{x}{2}-x\sqrt{4-x^2}+C$; D) $2\arcsin\frac{x}{2}-\frac{1}{2}x\sqrt{4-x^2}+C$;

4. Quyidagi $\int\frac{dx}{(x+1)\sqrt{x^2+2x+2}}$ integralda qanday almashtirish bajariladi?

A) $x+1=t^2$; B) $\sqrt{x^2+2x+1}=t$; C) $x+1=1/t$; D) $x=tgt$.

5. Quyidagi $\int\frac{dx}{x^2\sqrt{x^2-9}}$ integralda qanday almashtirish bajariladi?

A) $x=3/\cos t$; B) $x=3\sin t$; C) $x^2-9=t^2$; D) $x=3tgt$.

Shaxsiy uy topshiriqlari

Aniqmas integrallarni hisoblang.

1

1.1. a) $\int \frac{dx}{x\sqrt{x^2+1}}.$

b) $\int (4-3x)e^{-3x}dx.$

d) $\int \frac{12-6x}{(x+1)(x^2-4x+13)}dx$

1.2. a) $\int \frac{1+\ln x}{x}dx.$

b) $\int \operatorname{arctg}\sqrt{4x-1}dx.$

d) $\int \frac{x^3+6x^2+13x+8}{x(x+2)^3}dx.$

1.3. a) $\int \frac{dx}{x\sqrt{x^2-1}}.$

b) $\int (3x+4)e^{3x}dx.$

d) $\int \frac{x^3-6x^2+13x-6}{(x+2)(x-2)^3}dx.$

1.4. a) $\int \frac{x^2+\ln x^2}{x}dx.$

b) $\int (4x-2)\cos 2x dx.$

d) $\int \frac{2x^3-2x^2+5}{(x-1)^2(x^2+4)}dx$

1.5. a) $\int \frac{xdx}{\sqrt{x^4+x^2+1}}.$

b) $\int e^{-2x}(4x-3)dx.$

d) $\int \frac{x^3-6x^2+11x-10}{(x+2)(x-2)^3}dx.$

1.6. a) $\int \frac{(\arccos x)^3-1}{\sqrt{1-x^2}}dx.$

b) $\int (5x-2)e^{3x}dx.$

d) $\int \frac{x^3+6x^2+11x+7}{(x+1)(x+2)^3}dx.$

1.7. a) $\int \operatorname{tg} x \ln\cos x dx.$

b) $\int \frac{xdx}{\cos^2 x}.$

d) $\int \frac{x^3+8x-2}{x^2(x^2+4)}dx$

1.8. a) $\int \frac{\operatorname{tg}(x+1)}{\cos^2(x+1)}dx.$

b) $\int \ln(x^2+4)dx.$

d) $\int \frac{2x^3+x+1}{(x+1)x^3}dx.$

1.9. a) $\int \frac{x^3}{(x^2+1)^2}dx.$

b) $\int (2-4x)\sin 2x dx.$

d) $\int \frac{4x+2}{x^4+4x^2}dx$

1.10. a) $\int \frac{1-\cos x}{(x-\sin x)^2}dx.$

b) $\int \operatorname{arctg}\sqrt{6x-1}dx.$

d) $\int \frac{x^2-2x+4}{x^3(x^2+1)}dx$

1.11. a) $\int \frac{x\cos x+\sin x}{(x\sin x)^2}dx.$

b) $\int (4-16x)\sin 4x dx.$

d) $\int \frac{x^3+x+2}{(x+2)x^3}dx.$

1.12. a) $\int \frac{xdx}{\sqrt{x^4-x^2-1}}.$

b) $\int e^{-3x}(2-9x)dx.$

d) $\int \frac{2x+22}{(x+2)(x^2-2x+10)}dx$

1.13. a) $\int \frac{dx}{\cos^2 x\sqrt{tg^3 x}}$

b) $\int \operatorname{arctg}\sqrt{3x-1}dx.$

d) $\int \frac{x^3-3x^2+5}{x^3(x^2+1)}dx$

1.14. a) $\int \frac{1/(2\sqrt{x})+1}{(\sqrt{x}+x)^2}dx.$

b) $\int \operatorname{arctg}\sqrt{5x-1}dx.$

d) $\int \frac{6x}{x^3-1}dx$

1.15. a) $\int \frac{(x^2+1)dx}{(x^3+3x+1)^5}.$

b) $\int (5x+6)\cos 2xdx.$

d) $\int \frac{x^3+3x^2-12x+4}{(x-1)^2(x^2+1)}dx$

1.16. a) $\int \frac{4\operatorname{arctg} x - x}{1+x^2}dx.$

b) $\int (3x-2)\cos 5xdx.$

d) $\int \frac{2x^3+6x^2+7x+1}{(x-1)(x+1)^3}dx.$

1.17. a) $\int \frac{x-(\operatorname{arctg} x)^4}{1+x^2}dx.$

b) $\int (x\sqrt{2}-3)\cos 2xdx.$

d) $\int \frac{x^3-6x^2+10x-10}{(x+1)(x-2)^3}dx.$

1.18. a) $\int \frac{x+\cos x}{x^2+2\sin x}dx.$

b) $\int (4x+7)\cos 3xdx.$

d) $\int \frac{2x^3+6x^2+7x}{(x-2)(x+1)^3}dx.$

1.19. a) $\int \frac{2\cos x+3\sin x}{(2\sin x-3\cos x)^3}dx.$

b) $\int \ln(\cos x)dx$

d) $\int \frac{x^2-2x+4}{x^3(x^2+1)}dx$

1.20. a) $\int \frac{3x-\arccos 2x}{\sqrt{1-4x^2}}dx$

b) $\int arctg\sqrt{2x+1}dx$

d) $\int \frac{2x^3+6x^2+5x}{(x+2)(x+1)^3}dx.$

1.21. a) $\int \frac{x^3+x}{x^4+1}dx.$

b) $\int \frac{\ln(\cos x)dx}{\sin^2 dx}$

d) $\int \frac{2x^3+6x^2+7x+4}{(x+2)(x+1)^3}dx.$

1.22. a) $\int \frac{5x-(\arcsin 3x)^3}{\sqrt{1-9x^2}}dx$

b) $\int cos(\ln x)dx$

d) $\int \frac{x^3+6x^2+10x+10}{(x-1)(x+2)^3}dx.$

1.23. a) $\int \frac{x+\cos 2x}{\sqrt{x^2+\sin 2x}}dx$

b) $\int \frac{\ln(\sin x)dx}{\cos^2 dx}$

d) $\int \frac{x^3+6x^2+13x+6}{(x-2)(x+2)^3}dx.$

1.24. a) $\int \frac{3x\sin 3x-\cos 3x}{(x\cos 3x)^3}dx$

b) $\int \sin(\ln x)dx$

d) $\int \frac{x^3-6x^2+13x-8}{x(x-2)^3}dx.$

1.25. a) $\int \frac{\sqrt[3]{ctg^2 3x}}{\sin^2 3x}dx$

b) $\int xarctg\, 2xdx$

d) $\int \frac{x^3-6x^2+13x-7}{(x+1)(x-2)^3}dx.$

1.26. a) $\int \frac{3x-(arcctgx)^3}{\sqrt{1-x^2}}dx$

b) $\int x^2 arctgxdx$

d) $\int \frac{x^3+6x^2+14x+10}{(x+1)(x+2)^3}dx.$

1.27. a) $\int \frac{3x+x^3}{x^4+2}dx$

b) $\int (1-6x)e^{2x}dx.$

d) $\int \frac{2x^3+6x^2+7x+2}{x(x+1)^3}dx.$

1.28. a) $\int \frac{x^2-\ln^2 x}{x}dx$

b) $\int \frac{\arcsin x}{\sqrt{x+1}}dx$

d) $\int \frac{3x^3+9x^2+10x+2}{(x-1)(x+1)^3}dx.$

1.29. a) $\int \frac{dx}{(1+x^2)\sqrt{(arctgx)^3}}$

b) $\int xe^{-6x}dx$

d) $\int \frac{x^3-6x^2+14x-6}{(x+1)(x-2)^3}dx.$

1.30. a) $\int \frac{3\arcsin^2 x+4x}{\sqrt{1-x^2}}dx$

b) $\int (2x-5)\cos 4xdx.$

d) $\int \frac{2x^3+6x^2+5x+4}{(x-2)(x+1)^3}dx.$

2

2.1. a) $\int \frac{\sin^3 2x}{cos^2 2x}dx$

b) $\int tg^4 3xdx$

d) $\int \frac{dx}{3+cosx+\sin x}$

2.2. a) $\int \frac{\cos^3 x}{\sqrt[3]{cos^4 x}}dx$

b) $\int \sin^4 2xdx$

d) $\int \frac{dx}{3cos^2 x+4\sin^2 x}$

2.3. a) $\int \sin 4x\sin xdx$

b) $\int ctg^4 5xdx$

d) $\int \frac{dx}{2-3cosx+\sin x}$

2.4. a) $\int cos^4 3x\sin^2 3xdx$

b) $\int \sin^3 4xdx$

d) $\int \frac{dx}{4+3cosx-4\sin x}$

2.5. a) $\int \cos 4x\sin xdx$

b) $\int tg^3(4-x)dx$

d) $\int \frac{dx}{3+5\sin x+3cosx}$

2.6. a) $\int \sqrt[3]{\cos^4 x}\sin^3 xdx$

b) $\int tg^2(5x+1)dx$

d) $\int \frac{6\sin x+\cos x}{1+cosx}dx$

2.7. a) $\int \sqrt[3]{\sin^4 x}\cos^3 x dx$

b) $\int tg^5 4x dx$

d) $\int \frac{dx}{5-3cosx}$

2.8. a) $\int cos^3 2x \sin^3 2x dx$

b) $\int ctg^3 \frac{x}{2} dx$

d) $\int \frac{dx}{5+4\sin x}$

2.9. a) $\int cos^3 2x \sin^5 2x dx$

b) $\int \cos^4 3x dx$

d) $\int \frac{dx}{8+4cosx}$

2.10. a) $\int \cos x \sin 9x dx$

b) $\int \cos^3 4x dx$

d) $\int \frac{dx}{4\sin^2 x - 5cos^2 x}$

2.11. a) $\int \cos 2x \cos 5x dx$

b) $\int x tg^2 x^2 dx$

d) $\int \frac{dx}{8-4\sin x + 7\cos x}$

2.12. a) $\int \cos^4 x \sin x dx$

b) $\int (1-tg 2x)^2 dx$

d) $\int \frac{dx}{3+2cosx - \sin x}$

2.13. a) $\int \sin 5x \sin 7x dx$

b) $\int (1+ctg 2x)^2 dx$

d) $\int \frac{dx}{2\sin^2 x + 7\cos^2 x}$

2.14. a) $\int \sin^4 5x \cos 5x dx$

b) $\int (tg 2x + ctg 2x)^2 dx$

d) $\int \frac{dx}{8+7cosx - 4\sin x}$

2.15. a) $\int \frac{\cos^3 x}{\sqrt[3]{\sin^5 x}} dx$

b) $\int (1+\cos 3x)^2 dx$

d) $\int \frac{dx}{4\sin^2 x + 8\cos x \sin x}$

2.16. a) $\int \cos^4 x \sin 2x dx$

b) $\int ctg^2 \frac{x}{3} dx$

d) $\int \frac{dx}{3+3cosx + 2\sin x}$

2.17. a) $\int \cos^3 x \sin 2x dx$

b) $\int tg^4 \frac{x}{3} dx$

d) $\int \frac{dx}{5\sin^2 x - 3\cos^2 x}$

2.18. a) $\int \sin 5x cos 7x dx$

b) $\int tg^3 \frac{x}{2} dx$

d) $\int \frac{dx}{5+3cosx + \sin x}$

2.19. a) $\int \cos 5x \sin 7x dx$

b) $\int ctg^4 \frac{x}{2} dx$

d) $\int \frac{dx}{3+cosx + \sin x}$

2.20. a) $\int \cos^5 x \sin^3 x dx$

b) $\int tg^4 3x dx$

d) $\int \frac{dx}{16\sin^2 x + 7\cos^2 x}$

2.21. a) $\int \cos^2 x \sin^4 x dx$

b) $\int ctg^3(x+2)dx$

d) $\int \frac{dx}{7\sin x - 3cosx}$

2.22. a) $\int \sqrt[5]{\cos^4 x}\sin 2x dx$

b) $\int \cos^4(x+3)dx$

d) $\int \frac{dx}{4cosx - 6\sin x}$

2.23. a) $\int \cos^2 3x \sin^2 3x dx$

b) $\int (1 - tg3x)^2 dx$

d) $\int \frac{dx}{3 - 2\sin^2 x}$

2.24. a) $\int \cos^2 3x \sin^3 3x dx$

b) $\int (1 - \sin 3x)^2 dx$

d) $\int \frac{2 - \sin x + 3\cos x}{1 + cosx} dx$

2.25. a) $\int \cos 5x \cos 7x dx$

b) $\int tg^3(2x+3)dx$

d) $\int \frac{dx}{5 + 3\sin^2 x}$

2.26. a) $\int \cos^3 x \sin^7 x dx$

b) $\int (2 + \sin 5x)^2 dx$

d) $\int \frac{7 + 6\sin x - 5\cos x}{1 + cosx} dx$

2.27. a) $\int \cos 2x \sin^2 x dx$

b) $\int (tg3x - ctg3x)^2 dx$

d) $\int \frac{dx}{6 - 3\cos^2 x}$

2.28. a) $\int \sqrt[3]{\cos^2 3x}\sin 3x dx$

b) $\int ctg^3(2x-3)dx$

d) $\int \frac{dx}{2 + 3cosx + 4\sin x}$

2.29. a) $\int \sqrt[3]{\cos^2 x}\sin^3 x dx$

b) $\int tg^2 \frac{2x}{3} dx$

d) $\int \frac{\sin^2 x dx}{3\sin^2 x - \cos^2 x}$

2.30. a) $\int \cos 2x \cos 7x dx$

b) $\int tg^4(x+3)dx$

d) $\int \frac{dx}{8 - 3\sin^2 x}$

3

3.1. a) $\int \frac{\sqrt{x+1}dx}{\sqrt[3]{x+1} - \sqrt[6]{x+1}}$

b) $\int \frac{(x+3)dx}{\sqrt{x^2 - 2x + 6}}$

3.2. a) $\int \frac{xdx}{2 + \sqrt{x+4}}$

b) $\int \frac{dx}{x\sqrt{x^2 + x - 2}}$

3.3. a) $\int \frac{\sqrt[6]{x+2}dx}{\sqrt[3]{x+2} + \sqrt{x+2}}$

b) $\int \frac{dx}{\sqrt{(4 + x^2)^3}}$

3.4. a) $\int \frac{dx}{\sqrt[3]{(2x+3)^2}-\sqrt{2x+3}}$

b) $\int \frac{dx}{x^2\sqrt{x^2+25}}$

3.5. a) $\int \frac{(x-1)dx}{x\sqrt{x-2}}$

b) $\int \frac{dx}{x\sqrt{1+x-x^2}}$

3.6. a) $\int \frac{\sqrt{x+3}-\sqrt[3]{x+3}}{\sqrt[3]{(x+3)^2}+\sqrt[6]{x+3}}dx$

b) $\int \frac{(x-3)dx}{\sqrt{2x^2-4x+1}}$

3.7. a) $\int \frac{\sqrt[3]{(x+1)^2}+\sqrt[6]{x+1}}{\sqrt{x+1}+\sqrt[3]{x+1}}dx$

b) $\int \frac{(2x+1)dx}{\sqrt{1+x-3x^2}}$

3.8. a) $\int \frac{\sqrt[4]{x}+\sqrt{x}}{\sqrt{x}+1}dx$

b) $\int \frac{(2x-10)dx}{\sqrt{1+x-x^2}}$

3.9. a) $\int \frac{dx}{3+\sqrt{x+5}}$

b) $\int \frac{(2x+5)dx}{\sqrt{9+8x+4x^2}}$

3.10. a) $\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx$

b) $\int \frac{(3x+4)dx}{\sqrt{13+6x+x^2}}$

3.11. a) $\int \frac{\sqrt{x+1}-2\sqrt[3]{x-1}}{\sqrt{x+1}+2\sqrt[3]{x+1}}dx$

b) $\int \frac{(3x-1)dx}{\sqrt{2x^2-5x+1}}$

3.12. a) $\int \frac{(x+1)dx}{x\sqrt{x-1}}$

b) $\int \frac{(5x+2)dx}{\sqrt{x^2+3x-4}}$

3.13. a) $\int \frac{\sqrt{x+2}dx}{\sqrt[3]{x+2}+\sqrt[6]{x+2}}$

b) $\int \frac{(x-4)dx}{\sqrt{2x^2-x+7}}$

3.14. a) $\int \frac{\sqrt{x+3}dx}{1+\sqrt[3]{x+2}}$

b) $\int \frac{(4x+1)dx}{\sqrt{2+x-x^2}}$

3.15. a) $\int \frac{(x+3)dx}{x\sqrt{x-4}}$

b) $\int \frac{dx}{(x+1)\sqrt{1+x-x^2}}$

3.16. a) $\int \frac{\sqrt{x}+\sqrt[3]{x}}{\sqrt{x}+\sqrt[6]{x}}dx$

b) $\int \frac{(5x-3)dx}{\sqrt{2x^2+4x-5}}$

3.17. a) $\int \frac{\sqrt{3x+1}-2}{\sqrt{3x+1}+2\sqrt[3]{3x+1}}dx$

b) $\int \frac{(3x+2)dx}{\sqrt{4+2x-x^2}}$

3.18. a) $\int \frac{(x^3-1)dx}{\sqrt{x+2}}$

b) $\int \frac{dx}{x\sqrt{x^2+x-3}}$

3.19. a) $\int \frac{\sqrt{x}-\sqrt[3]{x}}{\sqrt[3]{x}-\sqrt[6]{x}-1}dx$

b) $\int \frac{(x+5)dx}{\sqrt{3-6x-x^2}}$

3.20. a) $\int \frac{\sqrt[6]{3x+1}+1}{\sqrt{3x+1}-\sqrt[3]{3x+1}}dx$

b) $\int \frac{(x-9)dx}{\sqrt{4+2x-x^2}}$

3.21. a) $\int \frac{x^3dx}{\sqrt{x-2}}$

b) $\int \frac{(3x-4)dx}{\sqrt{2x^2-6x+1}}$

3.22. a) $\int \frac{\sqrt{x}}{1-\sqrt[4]{x}}dx$

b) $\int \frac{(7x-1)dx}{\sqrt{2-3x-x^2}}$

3.23. a) $\int \frac{\sqrt{x}}{x-4\sqrt[3]{x^2}}dx$

b) $\int \frac{dx}{(x-1)\sqrt{1+x-x^2}}$

3.24. a) $\int \frac{x^2dx}{\sqrt{x-3}}$

b) $\int \frac{dx}{x\sqrt{1-x-x^2}}$

3.25. a) $\int \frac{x-\sqrt[3]{x^2}}{x(1+\sqrt[6]{x})}dx$

b) $\int \frac{dx}{x\sqrt{x^2-3x+2}}$

3.26. a) $\int \frac{\sqrt{x-2}dx}{3+\sqrt{x-2}}$

b) $\int \frac{(7x+1)dx}{\sqrt{2-4x-x^2}}$

3.27. a) $\int \frac{\sqrt{x}}{3x+\sqrt[3]{x^2}}dx$

b) $\int \frac{dx}{(x+1)\sqrt{x^2-3x+2}}$

3.28. a) $\int \frac{dx}{3+\sqrt{x-5}}$

b) $\int \frac{dx}{x\sqrt{1-3x-2x^2}}$

3.29. a) $\int \frac{dx}{2+\sqrt{x-8}}$

b) $\int \frac{dx}{(x+1)\sqrt{2-x-x^2}}$

3.30. a) $\int \frac{\sqrt{x}}{1-\sqrt[3]{x}}dx$

b) $\int \frac{(5x+1)dx}{\sqrt{3-6x-x^2}}$

6.5 Aniq integral va uni hisoblash

$y=f(x)$ funksiya $[a;b]$ kesmada aniqlangan bo‘lsin. Bu kesmani ixtiuoriy $a=x_0<x_1<x_2<...<x_n=b$ nuqtalar bilan *n* ta uzunliklari $\Delta x_i=x_i-x_{i-1}\left(i=\overline{1,n}\right)$ bo‘lgan qismiy bo‘laklarga bo‘lamiz va har bir bolakdan bittadan ixtiyoriy ξ_i nuqtalarni tanlab olamiz, bu yerda $x_{i-1}<\xi_i<x_i$ $i=\overline{1,n}$. Quyidagi ko‘rinishdagi yig‘indini tuzamiz:

$$\sigma_n=\sum_{i=1}^{n} f(\xi_i)\Delta x_i. \qquad (5.1)$$

Bu yig‘indi $y=f(x)$ funksiyaning $[a;b]$ kesmadagi *integral yig‘indisi* deyiladi. Integral yig‘indi asosi Δx_i balandligi $f(\xi_i)$ bo‘lgan to‘g‘ri to‘rtburchaklarning yuzalarining algebraik yig‘indisini beradi.

Integral yig‘indi σ_n ning qismiy bo‘laklar uzunliklarining eng kattasi nolga intilgandagi limiti $f(x)$ funksiyadan a dan b gacha olingan *aniq integral* deyiladi va $\int_a^b f(x)dx$ kabi belgilanadi, ya’ni ta’rif bo‘yicha

$$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \int_a^b f(x)dx \tag{5.2}$$

Teorema *Agar $f(x)$ funksiya $[a;b]$ kesmada uzluksiz bo‘lsa, u shu kesmada integrallanuvchidir, ya’ni bunday funksiya uchun (5.1) integral yig‘indining limiti mavjud va bu limit $[a;b]$ kesmani qismiy bo‘laklarga bo‘lish va ulardan ξ_i nuqtalarni tanlash usuliga bog‘liq emas.*

Agar $[a;b]$ kesmada $f(x) \geq 0$ bo‘lsa, $\int_a^b f(x)dx$ aniq integral $f(x)$ funksiya gafigi, Ox o‘qi va $x=a$, $x=b$ to‘g‘ri chiziqlar bilan chegaralangan *egri chiziqli trapetsiya yuzi*ni ifodalaydi.

1-Izoh. *Aniq integral integrallash o‘zgaruvchisiga bog‘liq emas:*

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(z)dz.$$

2-Izoh. *Aniq integralning chegaralari almashtirilsa, integralning ishorasi o‘zgaradi:*

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

3-Izoh. *Agar aniq integralning integrallash chegaralari teng bo‘lsa, uning qiymati nolga teng:*

$$\int_a^a f(x)dx = 0.$$

Aniq integralning asosiy xossalari ($f(x), \varphi(x)$ funksiyalarni mos kesmalarda integrallanuvchi deb faraz qilamiz):

1. *Bir nechta funksiyaning algebraik yig‘indisining aniq integrali qo‘shiluvchilar integrallarining yig‘indisiga teng. Ikki qo‘shiluvchi bo‘lgan hol bilan cheklanamiz:*

$$\int_a^b [f(x) \pm \varphi(x)]dx = \int_a^b f(x)dx \pm \int_a^b \varphi(x)dx.$$

2. *O'zgarmas ko'paytuvchini aniq integral belgisidan tashqariga chiqarish mumkin, agar* $k = const$ *bo'lsa, u holda*

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx.$$

3. *Agar* $[a;b]$ *kesmada funksiya o'z ishorasini o'zgartirmasa, u holda bu funksiya aniq integralining ishorasi funksiya ishorasi bilan bir xil bo'ladi, ya'ni:*

a) *agar* $[a;b]$ *kesmada* $f(x) \geq 0$ *bo'lsa, u holda*

$$\int_a^b f(x)dx \geq 0$$

b) agar $[a;b]$ *kesmada* $f(x) \leq 0$ *bo'lsa, u holda*

$$\int_a^b f(x)dx \leq 0$$

4. *Agar* $[a;b]$ *kesmada ikki* $f(x)$ *va* $\varphi(x)$ *funksiya* $f(x) \geq \varphi(x)$ *shartni qanoatlantirsa, u holda*

$$\int_a^b f(x)dx \geq \int_a^b \varphi(x)dx$$

5. *Agar* $[a;b]$ *kesma bir necha qismlarga bo'linsa, u holda* $[a;b]$ *kesma bo'yicha aniq integral har bir qism bo'yicha olingan aniq integrallar yig'indisiga teng.* $[a;b]$ *kesma ikki qismga bo'lingan hol bilangina cheklanamiz, ya'ni agar* $a < c < b$ *bo'lsa, u holda*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

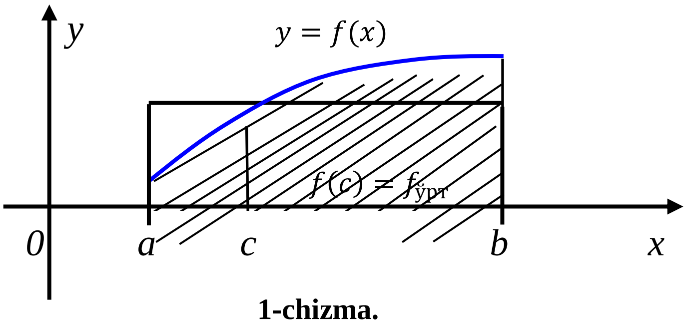
6. *Agar* m *va* M *sonlar* $f(x)$ *funksiyaning* $[a;b]$ *kesmadagi eng kichik va eng katta qiymatlari bo'lsa, u holda*

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

Teorema ***(o'rta qiymat haqida).*** *Agar* $f(x)$ *funksiya* $[a;b]$ *kesmada uzluksiz bo'lsa, bu kesmaning ichida shunday* $x = c$ *nuqta topiladiki,*

$$\int_a^b f(x)dx = f(c)(b-a)$$

Funksiyaning bu nuqtadagi qiymati uning shu kesmadagi o'rta qiymati bo'ladi.



1-chizma.

O'rta qiymat haqidagi teoremaning geometrik ma'nosi quyidagicha(1-chizma): yuqoridan integral osti funksiyasi $f(x)$ ning grafigi bilan chegaralangan, $(b-a)$ asosli egri chiziqli trapesiyaning yuzi o'shanday asosli va balandligi funksiyaning $f(c)$ o'rta qiymatiga teng to'g'ri to'rtburchakning yuziga tengdosh.

9. *Agar $f(x)$ funksiya kesmada uzluksiz va $\Phi(x) = \int_a^x f(x)dx$ bo'lsa, quyidagi tenglik o'rinli*

$$\Phi'(x) = \left(\int_a^x f(x)dx\right)' = f(x)$$

10. *Agar $F(x)$ funksiya $f(x)$ funksiyaning qandaydir boshlang'ich funksiyasi bo'lsa,*

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a) \qquad (5.3)$$

tenglik o'rinli. Bu formula *Nyuton-Leybnits formulasi* deyiladi.

1-misol

Aniq integralni hisoblang: $\int_1^4 \frac{x^2-2}{\sqrt{x}}dx$.

►Integral ostidagi funksiyani hadlab bo'lamiz va yuqoridagi xossalardan foydalanib integralni hisoblaymiz.

$$\int_1^4 \left(\frac{x^2}{\sqrt{x}} - \frac{2}{\sqrt{x}}\right)dx = \int_1^4 x^{3/2}dx - 2\int_1^4 \frac{1}{\sqrt{x}}dx = \frac{2}{5}x^{5/2}\Big|_1^4 - \sqrt{x}\Big|_1^4 = \frac{2}{5}(32-1) - (2-1) = 11\frac{2}{5}. ◄$$

2-misol

Aniq integralni hisoblang: $\int\limits_1^2 \frac{3x-4}{x^3+4x}dx$.

▶Integral ostidagi funksiyani sodda kasrlarga ajratamiz:

$$\frac{3x-4}{x^3+4x}=\frac{3x-4}{x(x^2+4)}=\frac{A}{x}+\frac{Bx+C}{x^2+4},\ A(x^2+4)+Bx^2+Cx\equiv 3x-4,$$

$$\begin{array}{c|c} x^2 & A+B=0 \\ x^1 & C=3 \\ x^0 & 4A=-4 \end{array}$$

Bundan, $A=-1, B=1, C=3$. Natijada,

$$\int\limits_1^2 \frac{2x-3}{x^3+4x}dx=\int\limits_1^2\left(-\frac{1}{x}+\frac{x+3}{x^2+4}\right)dx=-\int\limits_1^2\frac{dx}{x}+\int\limits_1^2\frac{xdx}{x^2+4}+3\int\limits_1^2\frac{dx}{x^2+4}=$$

$$=-\ln x\Big|_1^2+\frac{1}{2}\ln\left(x^2+4\right)\Big|_1^2+\frac{3}{2}arctg\frac{x}{2}\Big|_1^2=-\ln 2+\frac{1}{2}\ln\frac{8}{5}+\frac{3}{2}\left(arctg1-arctg\frac{1}{2}\right)=$$

$$=\ln\frac{\sqrt{10}}{5}+\frac{3}{2}arctg\frac{1}{3}.◀$$

$y=f(x)$ funksiya $[a;b]$ kesmada uzluksiz, $x=\varphi(t)$ funksiya hosilasi bilan $[\alpha;\beta]$ kesmada uzluksiz va monoton, $\varphi(\alpha)=a,\ \varphi(\beta)=b$ va $y=f(\varphi(t))$ murakkab funksiya $[\alpha;\beta]$da uzluksiz bo‘lsa, u holda quyidagi

$$\int\limits_a^b f(x)dx=\int\limits_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)dt \tag{5.4}$$

aniq integralda o‘zgaruvchini almashtirish formulasi o‘rinli.

3-misol

Aniq integralni hisoblang: $\int\limits_0^2\sqrt{4-x^2}dx$.

▶ $x=2\sin t$ deb almashtirish bajaramiz. U holda $dx=2\cos tdt$, $x=0$ da $\alpha=0$ va $x=2$ da $\beta=\pi/2$ ni hosil qilamiz. Natijada, (5.4) formulaga ko‘ra,

$$\int\limits_0^2\sqrt{4-x^2}dx=\int\limits_0^{\pi/2}\sqrt{4-4\sin^2x}\cdot 2\cos tdt=\int\limits_0^{\pi/2}4\cos^2tdt=2\int\limits_0^{\pi/2}(1+\cos 2t)dt=$$

$$=2\left(t+\frac{1}{2}\sin 2t\right)\Bigg|_0^{\pi/2}=\pi.◀$$

Agar $u(x)$ va $v(x)$ funksiyalar $[a;b]$ kesmada uzluksiz hosilalarga ega bo‘lsa, quyidagi

$$\int\limits_a^b u(x)dv(x)=u(x)v(x)\Big|_a^b-\int\limits_a^b v(x)du(x)$$

yoki qisqacha

$$\int_a^b u dv = uv\Big|_a^b - \int_a^b v du \qquad (5.5)$$

aniq integralda bo'laklab integrallash formulasi o'rinli.

4-misol

Aniq integralni hisoblang: $\int_1^e x^2 \ln x dx$.

► $\int_1^e x^2 \ln x dx = \left| \begin{array}{ll} u = \ln x, & du = \dfrac{dx}{x} \\ dv = x^2 dx, & v = x^3/3 \end{array} \right| = \dfrac{x^3}{3} \ln x \Big|_1^e - \dfrac{1}{3} \int_1^e x^2 dx = \dfrac{e^3}{3} - \dfrac{x^3}{9} \Big|_1^e = \dfrac{2e^3 + 1}{9}$. ◄

Auditoriya topshiriqlari

Berilgan aniq integrallarni hisoblang.

1. $\int_1^2 \frac{2x^6 + 2}{x^4} dx$.

2. $\int_{-1}^2 \frac{dx}{x^2 + 4x + 5}$.

3. $\int_1^{e^3} \frac{dx}{x\sqrt{1 + \ln x}}$.

4. $\int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x} dx$.

5. $\int_1^2 \frac{dx}{x\sqrt{x^2 + 5x + 1}}$

6. $\int_{-2}^2 \frac{dx}{1 + \sqrt{x + 2}}$

7. $\int_{\pi/3}^{\pi/2} \frac{dx}{\sin^3 x}$

8. $\int_0^3 \frac{dx}{(9 + x^2)\sqrt{9 + x^2}}$

9. $\int_1^{\sqrt{3}} arctg \frac{1}{x} dx$

10. $\int_0^{\pi/4} xtg^2 x dx$.

Mustaqil yechish uchun testlar

1. Aniq integralni hisoblang: $\int_0^1 \frac{x^3}{x^2 + 1} dx$

$A)\ \frac{1}{2} + \ln 2$ $\quad B)\ \frac{1}{2} + \ln \sqrt{2}$ $\quad C)\ \frac{1}{2} - \ln \sqrt{2}$ $\quad D)\ 1$

2. Aniq integralni hisoblang: $\int_0^{\pi/2} x \cos x dx$

$A)\ \frac{\pi}{2} - 1$ $\quad B)\ \frac{\pi}{2} + 1$ $\quad C)\ \frac{\pi}{2}$ $\quad D)\ 1$

3. Aniq integralni hisoblang: $\int_e^5 \frac{3}{x \ln x} dx$.

$A)\ 3\ln(\ln 5)$ $\quad B)\ 5\ln(\ln 3)$ $\quad C)\ 3\ln(5e)$ $\quad D)\ 5\ln(3e)$

4. Aniq integralni hisoblang: $\int_0^{\pi/4} \sin^3 2x dx$.

$A) -\frac{1}{3}$ $B)\ \frac{1}{3}$ $C)\ -\frac{2}{3}$ $D)\ \frac{2}{3}$

5. Aniq integralni hisoblang: $\int_3^4 \frac{dx}{x^2 - 3x + 2}$

$A) \ln\frac{2}{3}$ $B)\ \ln\frac{4}{3}$ $C)\ \ln\frac{3}{2}$ $D)\ \ln 2$

6.6 Aniq integralning tatbiqlari

6.6.1 Yassi shakllar yuzlarini hisoblash.

Yuqorida berilganidek, $[a;b]$ kesmada $f(x) \ge 0$ bo‘lsa aniq integral geometrik nuqtai nazardan egri chiziqli trapetsiyaning yuzini ifodalaydi. Ixtiyoriy yassi shaklni esa bir nechta egri chiziqli trapetsiyalar yuzlarining yig‘indisi yoki ayirmasi deb qarash mumkin. Bundan har qanday yassi shakllarning yuzini aniq integral yordamida hisoblash mumkinligi kelib chiqadi.

1-misol

Berilgan $y = x^2 - 2x$ funksiya grafigi, Ox koordinata chizig‘i va $x = -1,\ x = 1$ to‘g‘ri chiziqlar bilan chegaralangan shaklning yuzini hisoblang.

►Avval berilgan chiziqlar bilan chegaralangan yassi shaklni yasaymiz (1-chizma). Izlanayotgan yuza $S = |S_1| + |S_2|$ yoki $S = S_1 - S_2$ dan iborat,

$$S = \int_{-1}^{0} (x^2 - 2x)dx - \int_{0}^{1} (x^2 - 2x)dx =$$

$$= \left(\frac{x^3}{3} - x^2\right)\Bigg|_{-1}^{0} - \left(\frac{x^3}{3} - x^2\right)\Bigg|_{0}^{1} = -\left(-\frac{1}{3} - 1\right) - \left(\frac{1}{3} - 1\right) = 2. \blacktriangleleft$$

Umumiy holda, agar yassi shakl ikkita $y = f_1(x)$, $y = f_2(x)$ funksiyalar grafiklari va $x = a,\ x = b$ vertikal chiziqlar bilan chegaralangan bo‘lib, $f_1(x) \le f_2(x), x \in [a;b]$ bo‘lsa, bu yassi figura yuzi

$$S = \int_a^b (f_2(x) - f_1(x))dx \qquad (6.1)$$

formula bilan hisoblanadi.

Agar egri chiziqli trapetsiyaning chegarasidagi egri chiziq $x = x(t),\ y = y(t)$ parametrik tenglamalar bilan berilsa, u holda bu yassi shakl yuzi

$$S = \int_\alpha^\beta y(t)x'(t)dt \qquad (6.2)$$

formula bilan hisoblanadi. Bu yerda α va β chegara $a = x(\alpha), b = x(\beta)$ $(y(t) \ge 0, t \in [\alpha;\beta])$ tenglamalardan aniqlanadi.

2-misol

Ellips $\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\right)$ chizig‘i bilan chegaralangan shakl yuzuni toping.

►Avval ellipsning parametrik tenglamasini yozamiz: $x=a\cos t,\quad y=b\sin t$. Shaklning simmetrikligini va (6.2) formulani e’tiborga olib quyidagini hosil qilamiz:

$$S=4\int_0^a ydx=4\int_{\pi/2}^{0} b\sin t\cdot(-a\sin t)dt=4ab\int_0^{\pi/2}\sin^2 tdt=2ab(t-\frac{1}{2}\sin 2t)\Big|_0^{\pi/2}=\pi ab\,.◄$$

Egri chiziq qutb koordinatalar sistemasidagi $r=r(\varphi)$ tenglama bilan berilgan bo‘lsin. Agar OM_1M_2 yassi shakl $r=r(\varphi)$ tenglama bilan berilgan M_1M_2 egri chiziq va φ_1,φ_2 qutb burchaklariga mos keluvchi OM_1, OM_2 qutb radiuslari bilan chegaralangan egri chiziqli sektor bo‘lsa, uning yuzi

$$S=\frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2(\varphi)d\varphi \qquad (6.3)$$

formula bilan hisoblanadi.

3-misol

Qutb koordinatasida berilgan chiziq bilan chegaralangan shakl yuzini hisoblang: $r=4\cos 3\varphi$.

► $S=\dfrac{1}{2}\displaystyle\int_{\varphi_1}^{\varphi_2} r^2(\varphi)d\varphi$,

Qutb radiusi $r\geq 0$, ya’ni $4\cos 3\varphi\geq 0$, $\cos 3\varphi\geq 0$ bo‘ladi.

Bundan,

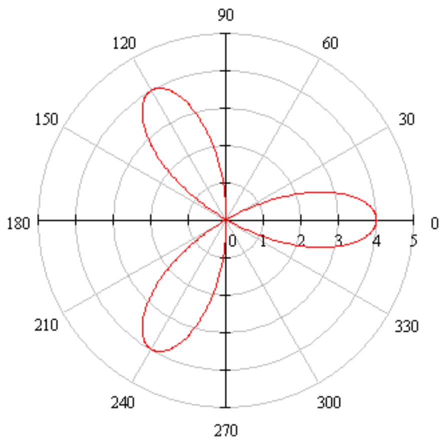
$$-\frac{\pi}{2}+2\pi n\leq 3\varphi\leq\frac{\pi}{2}+2\pi n, n\in Z\,,$$

$$-\frac{\pi}{6}+\frac{2\pi n}{3}\leq\varphi\leq\frac{\pi}{6}+\frac{2\pi n}{3}, n\in Z$$

Topilgan oraliqda $r=4\cos\varphi$ shaklni yasaymiz(2-chizma):

$$S=6\cdot\frac{1}{2}\int_{-\pi/6}^{0}16\cos^2 3\varphi d\varphi=24\int_{-\pi/6}^{0}(1+\cos 6\varphi)d\varphi=24(\varphi+\frac{1}{6}\sin 6\varphi)\Big|_{-\frac{\pi}{6}}^{0}=$$

$$=24(0+0+\frac{\pi}{6}+\frac{1}{6}\cdot 0)=4\pi.◄$$



2-chizma

6.6.2 Egri chiziq yoyi uzunligini hisoblash.

AB egri chiziq yoyi $y = f(x)$ tenglama bilan berilgan bo‘lsin, bu yerda $f(x)$ uzluksiz differensiyalanuvchi funksiya. U holda uning $l = AB$ uzunligi quyidagi formula bilan hisoblanadi:

$$l = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,dx \qquad (6.4)$$

Bu yerda $A(a;\ f(a))$ va $B(b;\ f(b))$ yoy uchlari bo‘ladi.

Agar silliq egri chiziq $x = x(t),\ y = y(t)$ tenglamalar bilan berilgan bo‘lib, $x(t),\ y(t)$ - uzluksiz differensiallanuvchi funksiyalar bo‘lsa, AB egri chiziq yoyi uzunligi l quyidagi formula bilan hisoblanadi:

$$l = \int_\alpha^\beta \sqrt{(x'(t))^2 + (y'(t))^2}\,dt \qquad (6.5)$$

Bu yerda α va β chegara t parametrning yoyning A va B chegaralariga mos keluvchi qiymatlaridir.

Agar silliq egri chiziq yoyi qutb koordinatalar sistemasidagi $r = r(\varphi)$ tenglama bilan berilgan bo‘lsa, u holda yoy uzunligi

$$l = \int_{\varphi_1}^{\varphi_2} \sqrt{r^2(\varphi) + (r'(\varphi))^2}\,d\varphi \qquad (6.6)$$

formula bilan hisoblanadi, bu yerda φ_1 va φ_2 yoyning A va B chegaralariga mos qutb burchaklaridir.

4-misol

Ushbu $y = \frac{2}{3}\sqrt{x^3}$ egri chiziqning $x_1 = 3,\ x_2 = 8$ absissali uchlari orasidagi yoyi uzunligini toping.

►(6.4)dan foydalanamiz.

$$l=\int_3^8\sqrt{1+\left(\sqrt{x}\right)^2}dx=\int_3^8\sqrt{1+x}dx=\frac{2}{3}(1+x)^{\frac{3}{2}}\Big|_3^8=\frac{2}{3}\left(9^{\frac{3}{2}}-4^{\frac{3}{2}}\right)=\frac{38}{3}=12\frac{2}{3}.◄$$

5-misol

Ushbu $x=a(t-\sin t),\ y=a(1-\cos t)$ sikloidaning 1-arkasi uzunligini toping.

► $x'_t=a(1-\cos t),\ y'_t=a\sin t$. Sikloidaning 1-arkasida $0\le t\le 2\pi$ ekanligidan va (6.5)dan foydalanib hisoblaymiz.

$$l=\int_0^{2\pi}\sqrt{a^2(1-\cos t)^2+a^2\sin^2 t}dt=a\int_0^{2\pi}\sqrt{2-2\cos t}dt=2a\int_0^{2\pi}\sin\frac{t}{2}dt=$$

$$=-4a\cos\frac{t}{2}\Big|_0^{2\pi}=-4a(-1-1)=8a.◄$$

6.6.3 Jism hajmini hisoblash.

Fazoda Ox o‘qiga proyeksiyasi $[a;\,b]$ kesma bo‘lgan qandaydir jism berilgan bo‘lsin. $x\in[a;\,b]$ nuqtadan o‘tuvchi Ox o‘qiga perpendikulyar har qanday tekislikning kesm bilan kesishmasi yuzi $S(x)$ ga teng bo‘lgan shaklni hosil qiladi. U holda bu jismning hajmi quyidagi formula bilan hisoblanadi:

$$V=\int_a^b S(x)dx \tag{6.7}$$

Xususiy holda, $y=f(x)$ funksiya gafigi bilan berilgan AB egri chiziq, Ox o‘qi va $x=a,\ x=b$ to‘g‘ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyani Ox o‘qi atrofida aylantirishdan hosil bo‘lgan jismning ko‘ndalang kesimi yuzi $S(x)=\pi f^2(x)$ bo‘ladi. Shuning uchun bu *aylanma jismning hajmi*

$$V=\pi\int_a^b f^2(x)dx \tag{6.8}$$

formula bilan hisoblanadi. Qisqacha, $V=\pi\int_a^b y^2dx$.

Xuddi shu kabi, yassi shaklni Oy o‘qi atrofida aylanishidan hosil bo‘lgan jism hajmini topish uchun $V=\pi\int_c^d x^2dy$ formula qo‘llanadi.

Eslatma. *Qutb koordinatalar sistemasining* $(r;\,\varphi)$ *o‘zgaruvchilari o‘rniga* $(\rho;\,\varphi)$ *o‘zgaruvchilarini ishlatish ham mumkin. U holda yuqoridagi (6.3) va (6.6) formulalar quyidagi ko‘rinishni oladi:*

$$S=\frac{1}{2}\int_{\varphi_1}^{\varphi_2}\rho^2(\varphi)d\varphi \quad \text{va} \quad l=\sqrt{\rho^2(\varphi)+\left(\rho'(\varphi)\right)^2}d\varphi$$

Auditoriya topshiriqlari

1. Berilgan egri chiziqlar bilan chegaralangan shakl yuzini toping:

$$a)\quad y^2 = x+5,\quad y^2 = -x+4$$

$$b)\quad y = (x-4)^2,\quad y = 16 - x^2$$

2. $x = a(t - \sin t),\ y = a(1 - \cos 3t)$ sikloidaning 1-arkasi va Ox o‘qi bilan chegaralangan shakl yuzini hisoblang.

3. $r = a(1 - \cos\varphi)$ kardioida chizig‘i bilan chegaralangan shakl yuzini toping.

4. $y = \frac{1}{3}\sqrt{(2x-1)^3}$ tenglama bilan berilgan egri chiziqning $x_1 = 2,\ x_2 = 8$ absissali uchlari orasidagi yoyi uzunligini hisoblang.

5. $x = a\cos t,\ y = b\sin t$ ellips chizig‘ining yoy uzunligini toping.

6. $r = a(1 - \cos\varphi)$ kardioida chizig‘i yoy uzunligini hisoblang.

7. $z = \frac{x^2}{4} + \frac{y^2}{2}$, $z = 1$ sirtlar bilan chegaralangan jism hajmini toping.

8. $x = a(t - \sin t),\ y = a(1 - \cos 3t)$ sikloidaning 1-arkasi va Ox o‘qi bilan chegaralangan shaklni absissa o‘qi atrofida aylanishidan hosil bo‘lgan jism hajmini toping.

Mustaqil yechish uchun testlar

1. Quyidagilardan qaysi biri yuzani topish formulasi emas?

A) $\int_\alpha^\beta y(t)x'(t)dt$ B) $\frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2(\varphi)d\varphi$ C) $\int_\alpha^\beta f^2(x)dx$ D) $\int_\alpha^\beta x(t)y'(t)dt$

2. Berilgan $f(x) = \sqrt{x^3}$ funksiyani $x = 0$ va $x = 5$ chiziqlar orasidagi yoyi uzunligini toping

A) 12 B) $12\frac{5}{27}$ C) $12\frac{7}{27}$ D) $12\frac{11}{27}$

3. Quyidagi $r^2 = 4\cos 2\varphi$ chiziq bilan chegaralangan figuraning yuzini hisoblang

A) 12 B) 8 C) 4 D) 16

4. $x = 4(t - \sin t),\quad y = 4(1 - \cos t),\quad 0 \le t \le \pi$ parametrik tenglama bilan berilgan egri chiziq yoy uzunligini hisoblang

A) 12 B) 8 C) 4 D) 16

5. Aniq integral yordamida hajm hisoblash formulasi berilgan variantni aniqlang

A) $\int_\alpha^\beta y(t)x'(t)dt$ B) $\frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2(\varphi)d\varphi$ C) $\int_\alpha^\beta f^2(x)dx$ D) $\int_\alpha^\beta x(t)y'(t)dt$

6.7 Birinchi va ikkinchi tur xosmas integrallar, ularni hisoblash va yaqinlashishga tekshirish

6.7.1 Chegarasi cheksiz xosmas integrallar.

Ta'rif. *Yarim* $[a,+\infty)$ *intervalda uzluksiz bo'lgan funksiyaning xosmas integrali quyidagicha belgilanadi:*

$$\int_a^{+\infty} f(x)dx$$

va ushbu tenglik bilan aniqlanadi:

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty}\int_a^b f(x)dx \tag{7.1}$$

Agar (7.1) formulada o'ngda turgan limit mavjud bo'lsa, u holda xosmas integral yaqinlashuvchi deyiladi. Bu limit integralning qiymati sifatida qabul qilinadi.

Agar ko'rsatilgan limit mavjud bo'lmasa, xosmas integral uzoqlashuvchi deb ataladi.

Agar integral ostidagi $f(x)$ funksiya uchun $F(x)$ boshlang'ich funksiya ma'lum bo'lsa, u holda xosmas integralning yaqinlashuvchimi yoki yo'qmi ekanini aniqlash mumkin. N'yuton-Leybnis formulalari yordamida quyidagiga ega bo'lamiz:

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty}\int_a^b f(x)dx = \lim_{b\to+\infty} F(x)\Big|_a^b = \lim_{b\to+\infty}[F(b)-F(a)] = F(+\infty)-F(a).$$

Shunday qilib, agar $x\to+\infty$ da $F(x)$ boshlang'ich funksiya ma'lum bo'lsa (biz uni $F(+\infty)$ bilan belgiladik), u holda xosmas integral yaqinlashuvchi, agar bu limit mavjud bo'lmasa, u holda xosmas integral uzoqlashuvchi bo'ladi.

1-misol

►Berilgan $f(x)=e^{-kx}$ funksiya uchun $F(x)=-\frac{1}{k}e^{-kx}$ funksiya boshlang'ich funksiya bo'ladi.

N'yuton-Leybnis formulasini qo'llaymiz:

$$I=\int_0^{+\infty} e^{-kx}dx = \lim_{b\to+\infty}\left(-\frac{1}{k}e^{-kx}\Big|_0^b\right) = -\frac{1}{k}\lim_{b\to+\infty}\left(e^{-kb}-1\right).$$

Agar $k>0$ bo'lsa, $I=\frac{1}{k}$ integral yaqinlashuvchi.

Agar $k\le 0$ bo'lsa, $I=\infty$ integral uzoq lashuvchi. ◄

Xosmas integral $(-\infty,b]$ yarim cheksiz integralda ham shunga o'xshash aniqlanadi:

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx = \lim_{a\to-\infty} F(x)\Big|_a^b = F(b)-F(-\infty).$$

bu yerda $F(-\infty)$ $F(x)$ boshlang'ich funksiyaning $x\to-\infty$ dagi limiti.

Agar $f(x)$ funksiya butun sonlar o'qida uzluksiz bo'lsa, u holda umumlashgan xosmas integral quyidagi formula bilan aniqlanadi:

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{s} f(x)\,dx + \int_{s}^{+\infty} f(x)\,dx \qquad (7.2)$$

bu yerda s –ixtiyoriy tayinlangan nuqta.

Agar (7.2) formulada o‘ng tomonda turgan ikkala integral yaqinlashuvchi bo‘lsa, u holda chap tomondagi xosmas integral ham yaqinlashuvchi bo‘ladi.

2-misol

Ushbu

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}.$$

integralni yaqinlashuvchiligini tekshiring.

► (7.2) formulada $s=0$ deb faraz qilib, quyidagini hosil qilamiz:

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2}.$$

Tenglikning o‘ng qismidagi xosmas integrallar yaqinlashuvchi bo‘ladi, chunki

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = arctgx\Big|_{-\infty}^{0} = arctg0 - arctg(-\infty) = \frac{\pi}{2},$$

$$\int_{0}^{+\infty} \frac{dx}{1+x^2} = arctgx\Big|_{0}^{+\infty} = arctg(+\infty) - arctg0 = \frac{\pi}{2}.$$

Shuning uchun ushbuga ega bo‘lamiz:

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Integarl yaqinlashuvchi va uning qiymati π ga teng. ◄

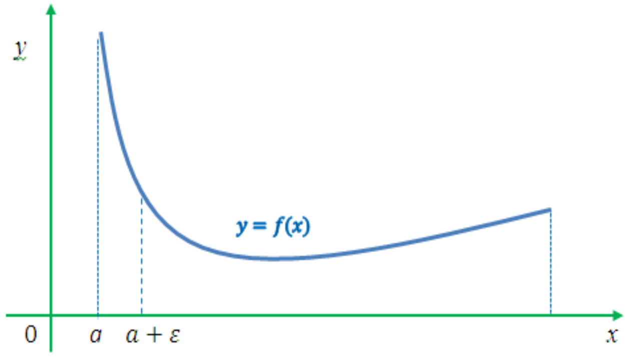
6.7.2 Cheksiz funksiyalarning xosmas integrallari.

Ta’rif. $(-\infty, b]$ *intervalda uzluksiz va* $x=a$ *da aniqlanmagan yoki uzilishga ega bo‘lgan* $f(x)$ *funksiyaning (1-shakl) xosmas integrali quyidagicha belgilanadi:*

$$\int_{a}^{b} f(x)\,dx$$

va ushbu tenglik bilan aniqlanadi:

$$\int_{a}^{b} f(x)\,dx = \lim_{\varepsilon \to +\infty} \int_{a+\varepsilon}^{b} f(x)\,dx \qquad (27.3)$$



1-chizma

Agar (7.3) formulada o‘ngda turgan limit mavjud bo‘lsa, u holda xosmas integral yaqinlashuvchi deyiladi.

Agar ko‘rsatilgan limit mavjud bo‘lmasa, u holda xosmas integral uzoqlashuvchi deyiladi.

Agar integral ostidagi $f(x)$ funksiya uchun $F(x)$ boshlang‘ich funksiya ma’lum bo‘lsa, u holda N’yuton-Leybnis formulasini qo‘llash mumkin:

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0} F(x)\Big|_{a+\varepsilon}^{b} = \lim_{\varepsilon\to 0}\left[F(b)-F(a+\varepsilon)\right] = F(b)-F(a)$$

Spunday qilib, agar $x \to a$ da $F(x)$ boshlang‘ich funksiyaning limiti mavjud bo‘lsa (biz uni $F(a)$ bilan belgiladik), u holda xosmas integral yaqinlashuvchi, agarda bu limit mavjud bo‘lmasa, u holda xosmas integral uzoqlashuvchi bo‘ladi.

$[a,\, b)$ intervalda uzluksiz va $x = b$ da aniqlanmagan yoki **II** tur uzilishga ega bo‘lgan $f(x)$ funksiyaning xosmas integrali ham shunga o‘xshash aniqlanadi:

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\int_a^{b-\varepsilon} f(x)dx = \lim_{\varepsilon\to 0} F(x)\Big|_a^{b-\varepsilon} = \lim_{\varepsilon\to 0}\left[F(b-\varepsilon)-F(a)\right] = F(b)-F(a),$$

bu yerda $F(b)-F(x)$ boshlang‘ich funksiyaning $x \to b$ dagi limiti.

Agarda $f(x)$ funksiya $[a,\, b]$ kesmaning biror-bir $x = s$ oraliq nuqtasida cheksiz uzilishga ega yoki aniqlanmagan bo‘lsa, u holda xosmas integral quyidagi integral bilan aniqlanadi:

$$\int_a^b f(x)dx = \int_a^s f(x)dx + \int_s^b f(x)dx \qquad (27.4)$$

Agar (7.4) formulaning o‘ng tomonida turgan intervalardan aqalli bittasi uzoqlashuvchi bo‘lsa, u holda xosmas integral uzoqlashuvchi bo‘ladiyu

Agar (7.4) ning o‘ng tomonidagi ikkala integral yaqinlashuvchi bo‘lsa, u holda tenglikning chap tomonidagi xosmas integral ham yaqinlashuvchi bo‘ladi.

3-misol

Ushbu

$$\int_0^4 \frac{dx}{\sqrt{x}}$$

integral ning yaqinlashuvchanligini tekshiring.

► $x \to 0$ da $f(x) = \frac{1}{\sqrt{x}} \to \infty$. $x = 0$ nuqta $[0,4]$ kesmaning chap oxirida yotadi. Shuning uchun quyidagiga ega bo‘lamiz:

$$\int_0^4 \frac{dx}{\sqrt{x}} = 2\sqrt{x} = 4 - 0 = 4.$$

Integral yaqinlashuvchi. ◄

6.7.3 Absolyut va shartli yaqinlashuvchanlik.

Ishorasini saqlamaydigan funksiyalarning xosmas integrallarini izlashni ba’zida nomanfiy funksiya bo‘lgan holga olib kelishga imkon beradigan alomatni keltiramiz.

Agar $\int_a^{+\infty} |f(x)| dx$ integral yaqinlashuvchi bo‘lsa, u holda $\int_a^{+\infty} f(x) dx$ integral ham yaqinlashuvchi bo‘ladi.

Bunda oxirgi integral ***absolyut*** yaqinlashuvchi interval deb ataladi.

Agarda $\int_a^{+\infty} f(x) dx$ integral yaqinlashuvchi, $\int_a^{+\infty} |f(x)| dx$ integral esa uzoqlashuvchi bo‘lsa, u holda $\int_a^{+\infty} f(x) dx$ integral ***shartli*** yaqinlashuvchi integral deb ataladi.

4-misol

Ushbu

$$\int_0^{+\infty} \frac{cosx}{1+x^2} dx, \qquad \int_0^{+\infty} \frac{sinx}{1+x^2} dx.$$

integrallarning yaqinlashuvchanligini tekshiring.

►Integral ostidagi funksiyalar ushbu shartlarni qanoatlantiradi:

$$\left| \frac{cosx}{1+x^2} \right| \le \frac{1}{1+x^2}, \qquad \left| \frac{sinx}{1+x^2} \right| \le \frac{1}{1+x^2}.$$

$$\int_0^{+\infty} \frac{dx}{1+x^2} = arctgx\Big|_0^{+\infty} = arctg(+\infty) - arctg0 = \frac{\pi}{2}$$

integral yaqinlashuvchi, shuning uchun

$$\int_0^{+\infty} \left| \frac{sinx}{1+x^2} \right| dx \qquad \int_0^{\infty} \left| \frac{cosx}{1+x^2} \right| dx$$

integrallar ham yaqinlashuvchi bo‘ladi. ◄

Auditoriya topshiriqlari

Xosmas integrallarni hisoblang yoki uzoqlashuvchi ekanini aniqlang.

1. $\int_e^{\infty}\frac{dx}{x(\ln x)^5}$ (Javob: $0,25$).

2. $\int_1^{\infty}\frac{dx}{\sqrt{x}}$ (Javob: Uzoqlashuvchi).

3. $\int_0^{\infty}x^5e^{-x^2}dx$ (Javob: 1).

4. $\int_{-\infty}^{\infty}\frac{2xdx}{x^2+1}$ (Javob: Uzoqlashuvchi).

5. $\int_{\sqrt{2}}^{\infty}\frac{dx}{x\sqrt{x^2-1}}$ (Javob: $\frac{\pi}{4}$).

6. $\int_0^{1}\frac{dx}{x^3}$ (Javob: Uzoqlashuvchi).

7. $\int_0^{1/e}\frac{dx}{x(\ln x)^2}$ (Javob: 1).

8. $\int_0^{2}\frac{dx}{x^2-4x+3}$ (Javob: Uzoqlashuvchi).

Xosmas integrallarni yaqinlashishga tekshiring.

9. $\int_{e^2}^{\infty}\frac{dx}{x\ln\ln x}$ (Javob: Uzoqlashuvchi).

10. $\int_0^{\pi/2}\frac{1-\cos x}{x^k}$ (Javob: $k\le 2$ da yaqinlashuvchi, $k>2$ da uzoqlashuvchi).

Mustaqil yechish uchun testlar

1. Quyidagi $\int_1^{\infty}\frac{2}{x^2}dx$ xosmas integralni hisoblang

A) 2 *B*) 1 *C*) 0 *D*) 3

2. Quyidagi $\int_0^{1}\frac{dx}{\sqrt[3]{x^2}}$ xosmas integralni hisoblang yoki uzoqlashuvchi ekanini aniqlang

A) 1 *B*) 2 *C*) 3 *D*) *uzoqlashuvchi*

3. Quyidagi $\int_1^{\infty}\frac{dx}{\sqrt[3]{x^2}}$ xosmas integralni hisoblang yoki uzoqlashuvchi ekanini aniqlang

$A)\ 1 \quad B)\ 2 \quad C)\ 3 \quad D)$ *uzoqlashuvchi*

4. Quyidagi $\int_0^2 \frac{dx}{x^2-3x+2}$ xosmas integralni hisoblang yoki uzoqlashuvchi ekanini aniqlang

$A)\ \ln 2 \quad B)\ -\ln 4 \quad C)\ -\ln 2 \quad D)$ *uzoqlashuvchi*

5. Quyidagi $\int_0^4 \frac{dx}{\sqrt{4-x}}$ xosmas integralni hisoblang yoki uzoqlashuvchi ekanini aniqlang

$A)\ 1 \quad B)\ 2 \quad C)\ 3 \quad D)$ *uzoqlashuvchi*

Shaxsiy topshiriqlar

1-topshiriq. Aniq integrallarni hisoblang.

1.1 a) $\int_1^{\sqrt{2}} \frac{\sqrt{4-x^2}}{x^2}dx$ b) $\int_0^{\pi/4} \frac{xdx}{\cos^2 x}$

1.2 a) $\int_1^2 \frac{\sqrt{x^2-1}}{x}dx$ b) $\int_0^2 arctg\sqrt{x+1}dx$

1.3 a) $\int_0^{\pi/4} \frac{dx}{\cos^2 x+3\sin^2 x}$ b) $\int_1^3 \ln(3x+2)dx$

1.4 a) $\int_0^{\ln 3} \frac{dx}{e^x(1+3e^{-2x})}$ b) $\int_0^1 \frac{\arcsin x}{\sqrt{x+1}}dx$

1.5 a) $\int_0^{\pi/2} \frac{dx}{\cos x-3\sin x}$ b) $\int_1^2 x^2\ln xdx$

1.6 a) $\int_{e^2}^{e^3} \frac{\ln xdx}{x(1+\ln^2 x)}$ b) $\int_0^1 \frac{\arccos x}{\sqrt{x+1}}dx$

1.7 a) $\int_0^4 x^2\sqrt{16-x^2}dx$ b) $\int_0^{\pi/4} xtg^2xdx$

1.8 a) $\int_{\pi/6}^{\pi/2} \frac{3\cos^3 x}{\sin^4 x}dx$ b) $\int_0^1 \left(x^2+x\right)e^x dx$

1.9 a) $\int_0^{\pi/2} \sin^4 x\cos^5 xdx$ b) $\int_1^2 (x-1)\ln xdx$

1.10 a) $\int_0^{\sqrt{5}} \sqrt{5-x^2}dx$ b) $\int_1^{\sqrt{3}} arctg\frac{1}{x}dx$

1.11 a) $\int_{-1}^{1}\frac{xdx}{\sqrt{5x-1}}$ b) $\int_{0}^{\pi/2} x\sin x\cos xdx$

1.12 a) $\int_{1/\sqrt{3}}^{1}\frac{dx}{x^2\sqrt{1+x^2}}$ b) $\int_{0}^{\pi/4} x\sin^2 xdx$

1.13 a) $\int_{4}^{9}\frac{\sqrt{x}dx}{\sqrt{x}-1}$ b) $\int_{0}^{\pi} x^2\sin xdx$

1.14 a) $\int_{0}^{4}\frac{dx}{1+\sqrt{2x+1}}$ b) $\int_{-2}^{0} x^2e^{-\frac{x}{2}}dx$

1.15 a) $\int_{\ln 2}^{\ln\sqrt{2}}\frac{dx}{e^x\sqrt{1-e^{-2x}}}$ b) $\int_{2/3}^{1} arctg(3x-2)dx$

1.16 a) $\int_{0}^{\pi/3}\frac{dx}{2+\cos x}$ b) $\int_{1}^{e^2}\sqrt{x}\ln xdx$

1.17 a) $\int_{3}^{6}\frac{\sqrt{x^2-9}}{x^4}dx$ b) $\int_{0}^{1}\frac{\ln(x+2)}{(x+2)^2}dx$

1.18 a) $\int_{2}^{4}\frac{dx}{x\sqrt{x-1}}$ b) $\int_{0}^{1}\frac{\arcsin(x/2)}{\sqrt{2-x}}dx$

1.19 a) $\int_{-2}^{2} x^2\sqrt{4-x^2}dx$ b) $\int_{0}^{\pi/6}\frac{xdx}{\cos^2 2x}$

1.20 a) $\int_{1}^{13}\frac{x+1}{\sqrt[3]{2x+1}}dx$ b) $\int_{1/2}^{1} arc\sin(1-x)dx$

1.21 a) $\int_{0}^{5}\frac{dx}{2x+\sqrt{3x+1}}$ b) $\int_{\pi/6}^{\pi/4} xctg^2xdx$

1.22 a) $\int_{3}^{6}\frac{\sqrt{x^2-9}}{x^4}dx$ b) $\int_{1}^{e}(x+3)\ln^2 xdx$

1.23 a) $\int_{1}^{2}\frac{\sqrt{x^2-1}}{x}dx$ b) $\int_{1}^{e} x\ln^2 xdx$

1.24 a) $\int_{0}^{\sqrt{6}}\sqrt{6-x^2}dx$ b) $\int_{1}^{3}\ln(2x+3)dx$

1.25

2-topshiriq. Quyidagi tenglama bilan berilgan chiziqlar bilan chegaralangan figura yuzini hisoblang.

2.1. $r = 4\sin^2\varphi$

2.2. $x = 3(\cos t + t\sin t),\ y = 3(\sin t - t\cos t), y = 0\ (0 \le t \le \pi)$

2.3. $r = 3\sin 4\varphi$

2.4. $r = 4\cos 3\varphi,\ r = 2\ (r \ge 2)$

2.5. $x = 4(t - \sin t),\ y = 4(1 - \cos t)$

2.6. $y = x + 1, y = \cos x,\ y = 0$

2.7. $r = 6\sin 3\varphi,\ r = 3\ (r \ge 3)$

2.8. $r = \cos\varphi + \sin\varphi$

2.9. $r = 1/2 + \sin\varphi$

2.10. $r = 6\cos 3\varphi,\ r = 3\ (r \ge 3)$

2.11. $r = \cos 3\varphi$

2.12. $r = \sin\varphi,\ r = 2\sin\varphi$

2.13. $r = 1/2 + \cos\varphi$

2.14. $x = 7\cos^3 t,\ y = 7\sin^3 t$

2.15. $y^2 = x^3, x = 4, y = 0$

2.16. $r = \cos\varphi - \sin\varphi$

2.17. $r = 1 + \sqrt{2}\sin\varphi$

2.18. $r = 5(1 - \cos\varphi)$

2.19. $r = \sqrt{3}\cos\varphi,\ r = \sin\varphi, 0 \le \varphi \le \pi/2$

2.20. $x = 5\cos^3 t,\ y = 5\sin^3 t$

2.21. $r = 3(1 - \cos\varphi)$

2.22. $r = 6\cos 3\varphi,\ r = 3\ (r \le 3)$

2.23. $r = \sin 3\varphi$

2.24. $r = 2(1 - \cos\varphi), r = 2(r \ge 2)$

2.25. $r = 5(1 + \cos\varphi)$

3-topshiriq. Quyidagi tenglamalar orqali berilgan chiziqning yoyi uzunligini hisoblang.

3.1. $r = 5(1 + \cos\varphi)$

3.2.
$$\begin{cases} x = 3(2\cos t - \cos 2t), \\ y = 3(2\sin t - \sin 2t), \end{cases}$$
$$0 \le t \le 2\pi.$$

3.3. $y = 1 + \ln(\cos x),\ (0 \le x \le \pi/6)$

3.4. $$\begin{cases} x = 4(\cos t + t\sin t), \\ y = 4(\sin t - t\cos t), \end{cases}$$
$$0 \le t \le 2\pi.$$

3.5. $$\begin{cases} x = (t^2 - 2)\sin t + 2t\cos t, \\ y = (2 - t^2)\cos t + 2t\sin t, \end{cases}$$
$$0 \le t \le \pi.$$

3.6. $$\begin{cases} x = 2\sin^3 t \\ y = 2\cos^3 t \end{cases} \quad 0 \le t \le \pi$$

3.7. $$\begin{cases} x = e^t(\cos t + \sin t), \\ y = e^t(\cos t - \sin t), \end{cases}$$
$$0 \le t \le \pi.$$

3.8. $$\begin{cases} x = 4\sin^3 t \\ y = 4\cos^3 t \end{cases}$$

3.9. $$\begin{cases} x = 3{,}5(2\cos t - \cos 2t), \\ y = 3{,}5(2\sin t - \sin 2t), \end{cases}$$
$$0 \le t \le \pi/2.$$

3.10. $$\begin{cases} x = 6\cos^3 t, \\ y = 6\sin^3 t, \end{cases}$$
$$0 \le t \le \pi/3.$$

3.11. $$\begin{cases} x = (t^2 - 2)\sin t + 2t\cos t, \\ y = (2 - t^2)\cos t + 2t\sin t, \end{cases}$$
$$0 \le t \le \pi/3.$$

3.12. $$\begin{cases} x = 8(\cos t + t\sin t), \\ y = 8(\sin t - t\cos t), \end{cases}$$
$$0 \le t \le \pi/4.$$

3.13. $r = 2\sin^3(\varphi/3),\ (0 \le \varphi \le \pi/6)$

3.14. $$\begin{cases} x = 3(\cos t + t\sin t), \\ y = 3(\sin t - t\cos t), \end{cases}$$
$$0 \le t \le \pi/3.$$

3.15. $$\begin{cases} x = 3(t - \sin t), \\ y = 3(1 - \cos t), \end{cases}$$
$$\pi \le t \le 2\pi.$$

3.16. $\begin{cases} x = e^t(\cos t + \sin t), \\ y = e^t(\cos t - \sin t), \end{cases}$

$\pi/2 \le t \le \pi.$

3.17. $\begin{cases} x = 2{,}5(t - \sin t), \\ y = 2{,}5(1 - \cos t), \end{cases}$

$\pi/2 \le t \le \pi.$

3.18. $\begin{cases} x = 3{,}5(2\cos t - \cos 2t), \\ y = 3{,}5(2\sin t - \sin 2t), \end{cases}$

$0 \le t \le \pi/2.$

3.19. $\begin{cases} x = 6(\cos t + t\sin t), \\ y = 6(\sin t - t\cos t), \end{cases}$

$0 \le t \le \pi.$

3.20. $\begin{cases} x = (t^2 - 2)\sin t + 2t\cos t, \\ y = (2 - t^2)\cos t + 2t\sin t, \end{cases}$

$0 \le t \le \pi/2.$

3.21. $\begin{cases} x = 8\cos^3 t, \\ y = 8\sin^3 t, \end{cases}$

$0 \le t \le \pi/6.$

3.22. $\begin{cases} x = (t^2 - 2)\sin t + 2t\cos t, \\ y = (2 - t^2)\cos t + 2t\sin t, \end{cases}$

$0 \le t \le 2\pi.$

3.23. $\begin{cases} x = 4(t - \sin t), \\ y = 4(1 - \cos t), \end{cases}$

$\pi/2 \le t \le 2\pi/3.$

3.24. $y^2 = x^3$ ning $x = 4$ bilan kesilgan qismi.

3.25. $\begin{cases} x = 5\sin^3 t \\ y = 5\cos^3 t \end{cases}$ $0 \le t \le \pi$

4-topshiriq. Quyidagi chiziqlar bilan chegaralangan figuraning Ox o‘qi(1-12 variantlar uchun), Oy o‘qi(13-25 variantlar uchun) atrofida aylanishidan hosil bo‘lgan jism hajmini toping.

4.1. $y^2 = x^3, x = 0, y = 4$

4.2. $x = 4(t - \sin t),\ y = 4(1 - \cos t), y = 0$

4.3. $y = 5\cos x, y = \cos x, x = 0, x \ge 0$

4.4. $y=2x-x^2, y=4x-2x^2$

4.5. $y=\sin^2 x, x=\pi/2, y=0$

4.6. $x=\sqrt[3]{y-2}\ x=1, y=1$

4.7. $y=xe^x, y=0,\ x=1$

4.8. $y=2x-x^2, y=-x+2, x=0$

4.9. $y=3\sin x, y=\sin x, 0\le x\le\pi$

4.10. $x=3\cos^2 t,\ y=2\sin^2 t$

4.11. $y=\arccos x,\ y=\arcsin x, x=0$

4.12. $(y-1)^2=x,\ y=x-1$

4.13. $x=2\cos^3 t,\ y=2\sin^3 t$

4.14. $y=\arccos x,\ y=\arcsin x, y=0$

4.15. $y=(x-1)^2,\ y=1\ y=(x-1)^2,\ \ y=1.$

4.16. $y^2=x-2,\ y=x^3,\ y=0, y=1$

4.17. $y=x^3, y=x^2$

4.18. $y=\arccos(x/5),\ y=\arcsin(x/3), y=0$

4.19. $(y-1)^2=x,\ y=x-1$

4.20. $y=(x-2)^2,\ y=4-x$

4.21. $y=\arccos x,\ y=\arcsin x, x=0$

4.22. $y=(x-1)^2,\ x=0,\ x=3,\ y=0$

4.23. $x=2\cos^2 t,\ y=5\sin^2 t$

4.24. $y=x^3, y=x$

4.25. $y=(x-1)^2,\ x=3,\ y=0$

5-topshiriq. Xosmas integrallarni hisoblang yoki uzoqlashuvchi ekanini isbotlang.

5.1. a) $\int_0^{\infty}\frac{xdx}{16x^2+1}$ b) $\int_0^1\frac{dx}{\sqrt{2-4x}}$

5.2. $\int_1^{\infty}\frac{16xdx}{16x^4-1}$ b) $\int_{-1}^3\frac{2x-3}{\sqrt[3]{x^2}}dx$

5.3. a) $\int_0^{\infty}\frac{x^3dx}{\sqrt{16x^4+1}}$ b) $\int_0^1\frac{e^{1/x}}{x^2}dx$

5.4. a) $\int_1^{\infty}\frac{xdx}{\sqrt{16x^4-1}}$ b) $\int_1^3\frac{dx}{\sqrt[3]{2-4x}}$

5.5. a) $\int_{-\infty}^0\frac{xdx}{\sqrt{(x^2+1)^3}}$ b) $\int_{1/2}^2\frac{\ln(2x-1)}{2x-1}dx$

5.6. a) $\int_0^{\infty} \frac{x^2dx}{\sqrt[3]{\left(x^3+1\right)^4}}$ b) $\int_{1/4}^{1} \frac{dx}{20x^2-9x+1}$

5.7. a) $\int_0^{\infty} \frac{xdx}{\sqrt[4]{\left(x^2+16\right)^5}}$ b) $\int_{1/2}^{1} \frac{dx}{(1-x)\ln^2(1-x)}$

5.8. a) $\int_4^{\infty} \frac{xdx}{\sqrt{x^2-4x+1}}$ b) $\int_0^{2/3} \frac{\sqrt[3]{\ln(2-3x)}}{2-3x}dx$

5.9. a) $\int_1^{\infty} \frac{xdx}{x^2-4x+5}$ b) $\int_0^{1} \frac{xdx}{1-x^4}$

5.10. a) $\int_{-1}^{\infty} \frac{dx}{x^2+4x+5}$ b) $\int_0^{\pi/6} \frac{\cos 3x}{\sqrt[3]{(1-\sin 3x)^2}}dx$

5.11. a) $\int_1^{\infty} \frac{arctg\, 2x}{4x^2+5}dx$ b) $\int_0^{1} \frac{2xdx}{\sqrt{1-x^4}}$

5.12. a) $\int_0^{\infty} \frac{dx}{4x^2+4x+5}$ b) $\int_0^{1/3} \frac{\cos xdx}{\sqrt[7]{\sin^2 x}}$

5.13. a) $\int_0^{\infty} \frac{xdx}{9x^2+6x+5}$ b) $\int_{4/5}^{1} \frac{5dx}{\sqrt[3]{4-5x}}$

5.14. a) $\int_0^{\infty} \frac{(x+3)dx}{\sqrt[3]{x^2+6x+5}}$ b) $\int_0^{\pi/2} \frac{e^{tgx}dx}{\cos^2 x}$

5.15. a) $\int_0^{\infty} \frac{3-x^2}{x^2+4}dx$ b) $\int_0^{1} \frac{e^{\pi-\arcsin x}dx}{\pi\sqrt{1-x^2}}$

5.16. a) $\int_0^{\infty} \frac{3+\sqrt{arctg\, 3x}dx}{9x^2+1}$ b) $\int_1^{2} \frac{3dx}{\sqrt[3]{4x-x^{2.}-4}}$

5.17. a) $\int_1^{\infty} \frac{4dx}{x(1+\ln^2 x)}$ b) $\int_{\pi/2}^{\pi} \frac{\sin 2xdx}{\sqrt[3]{1-\cos^2 x}}$

5.18. a) $\int_0^{\infty} x\sin xdx$ b) $\int_0^{1/3} \frac{5dx}{\sqrt[3]{1-3x}}$

5.19. a) $\int_{-\infty}^{-1} \frac{5dx}{\left(x^2-4x\right)\ln 3}$ b) $\int_1^{3} \frac{xdx}{\sqrt[3]{\left(x^2-1\right)^4}}$

5.20. a) $\int_0^{\infty} \frac{\pi dx}{(1+4x^2)arctg^2 2x}$ b) $\int_0^{1/3} \frac{dx}{9x^2-9x+2}$

5.21. a) $\int_1^{\infty} \frac{\left(2x+1\right)dx}{\sqrt{4x^2+4x+5}}$ b) $\int_0^{\pi/2} \frac{3\sin^3 xdx}{\sqrt{\cos x}}$

5.22. a) $\int_0^{\infty} \frac{dx}{\left(x^2+2x\right)\ln 5}$ b) $\int_0^{1} \frac{x^4dx}{\sqrt[3]{1-x^5}}$

5.23. a) $\int_{0}^{\infty} 2xe^{-3x}dx$ b) $\int_{0}^{\sqrt{5}} \frac{\sqrt[3]{5}xdx}{\sqrt[3]{5-x^2}}$

5.24. a) $\int_{-\infty}^{0} 3xe^{2x}dx$ b) $\int_{0}^{3} \frac{2x^2dx}{\sqrt{9-x^6}}$

5.25. a) $\int_{0}^{\infty} x^3e^{-x^2}dx$ b) $\int_{0}^{1/5} \frac{3xdx}{\sqrt[5]{1-25x^2}}$

Foydalanilgan asosiy darsliklar va o‘quv qo‘llanmalar ro‘yxati

1. Claudio Canuto, Anita Tabacco “Mathematical Analysis”, Italy, Springer, I-part, 2008, II-part, 2010.
2. W W L Chen “LINEAR ALGEBRA”, London, Chapter 1-12, 1983, 2008.
3. W W L Chen “Introduction to Fourier Series”, London, Chapter 1-8, 2004, 2013.
4. W W L Chen “Fundamentales of Analysis”, London, Chapter 1-10, 1983, 2008.
5. Жўраев Т., Саъдуллаев А., Худойберганов Г., Мансуров Х., Ворисов А. Олий математика асослари. Т.1., Тошкент, “Ўқитувчи”, 1995.
6. Жўраев Т., Саъдуллаев А., Худойберганов Г., Мансуров Х., Ворисов А. Олий математика асослари. Т.2., Тошкент, “Ўзбекистон”, 1999.
7. Соатов Ё.У, Олий математика. Т., Ўқитувчи, 1995. 1- 5 қисмлар.
8. N.M.Jabborov, E.«Oliy matematika». 1-2 qism. Qarshi, 2010.
9. Латипов Х.Р., Таджиев Ш. Аналитик геометрия ва чизиқли алгебра. Тошкент, "Ўзбекистон". 1995.
10. Азларов Т., Мансуров Х. Математик анализ, - Тошкент.: Ўқитувчи, 1-қисм, 1989.
11. Латипов Х.Р., Носиров Ф.У., Таджиев Ш.А. Аналитик геометрия ва чизиқли алгебрадан масалалар ечиш бўйича қўлланма. Тошкент, Фан, 1999.

Internet saytlari

1. www.Ziyonet.uz
2. www.tuit.uz
3. www.Math.uz
4. www.bilim.uz
5. www.gov.uz

"Oliy matematika.Ehtimollar nazariyasi va matematik statistika"
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