

Using Mathcad for Statics and Dynamics

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Hibbeler Bedford and Fowler

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Statics
Dynamics

1

Resolving Forces, Calculating Resultants

Ref: Hibbeler § 2.4-2.6, Bedford & Fowler: Statics § 2.3-2.5

Resolving forces refers to the process of finding two or more forces which, when combined, will produce a force with the same magnitude and direction as the original. The most common use of the process is finding the components of the original force in the Cartesian coordinate directions: x , y , and z .

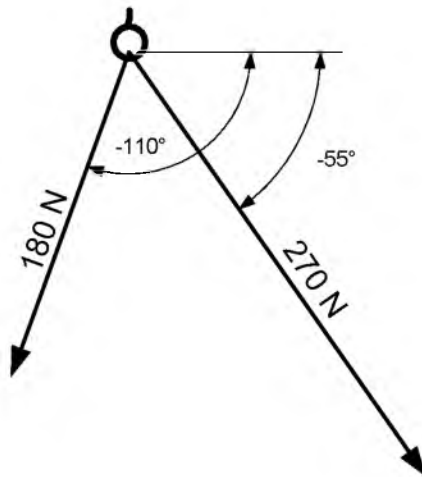
A *resultant* force is the force (magnitude and direction) obtained when two or more forces are combined (i.e., added as vectors).

Breaking down a force into its Cartesian coordinate components (e.g., F_x , F_y) and using Cartesian components to determine the force and direction of a resultant force are common tasks when solving statics problems. These will be demonstrated here using a two-dimensional problem involving co-planar forces.

Example: Co-Planar Forces

Two boys are playing by pulling on ropes connected to a hook in a rafter. The bigger one pulls on the rope with a force of 270 N (about 60 lb_f) at an angle of 55° from horizontal. The smaller boy pulls with a force of 180 N (about 40 lb_f) at an angle of 110° from horizontal.

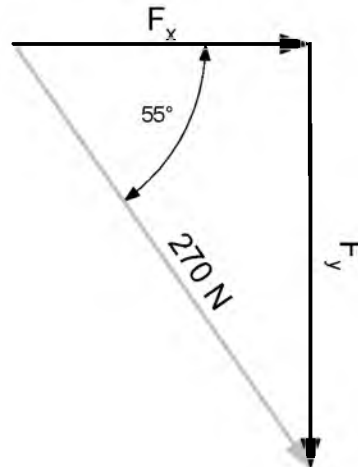
- Which boy is exerting the greatest vertical force (downward) on the hook?
- What is the net force (magnitude and direction) on the hook – that is, calculate the resultant force.



Note: The angles in this figure have been indicated as *coordinate direction angles*. That is, each angle has been measured from the positive x axis.

Solution

First, consider the 270 N force acting at 55° from horizontal. The x - and y -components of force are indicated schematically, as



The x- and y-components of the first force (270 N) can be calculated using a little trigonometry involving the included angle, 55°:

$$\cos(55^\circ) = \frac{F_{x1}}{270 \text{ N}}, \text{ or } F_{x1} = (270 \text{ N}) \cos(55^\circ)$$

and

$$\sin(55^\circ) = \frac{F_{y1}}{270 \text{ N}}, \text{ or } F_{y1} = (270 \text{ N}) \sin(55^\circ).$$

Mathcad can be used to solve for F_{x1} and F_{y1} using its built-in $\sin()$ and $\cos()$ functions, but these functions assume that the angle will be expressed as radians, not degrees. You can use the deg unit to explicitly tell Mathcad that the angle is in degrees and must be converted (by Mathcad) to radians.

$$F_{x1} := (270 \text{ N}) \cos(55 \text{ deg})$$

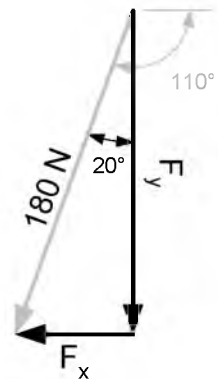
$$F_{x1} = 154.866 \text{ N}$$

$$F_{y1} := (270 \text{ N}) \sin(55 \text{ deg})$$

$$F_{y1} = 221.171 \text{ N}$$

Your Turn

Show that the x- and y-components of the second force (180 N acting at 110° from the x-axis) are 61.5 N (-x direction) and 169 N (-y direction), respectively. Note that trigonometry relationships are based on the included angle of the triangle (20°, as shown at the right), not the coordinate angle (-110° from the x-axis).

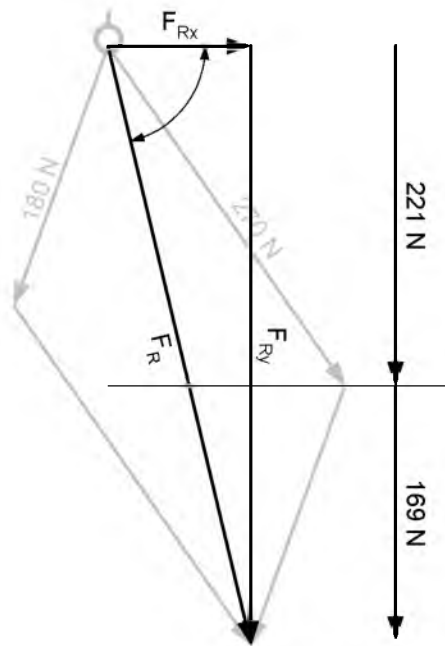


Answer, part a)

The larger boy exerts the greatest vertical force (221 N) on the hook. The vertical force exerted by the smaller boy is only 169 N.

Solution, continued

To determine the combined force on the hook, F_R , first add the two y-components calculated above, to determine the combined y-directed force, F_{Ry} , on the hook:

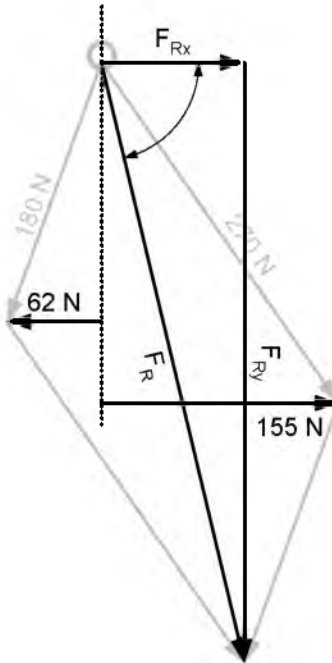


$$F_{Ry} := F_{y1} + F_{y2}$$

$$F_{Ry} = 390.316\text{N}$$

The y-component of the resultant force is 390 N (directed down, or in the $-y$ direction). Note that the direction has not been accounted for in this calculation.

Then add the two x-components to determine the combined x-directed force, F_{Rx} , on the hook. Note that the two x-component forces are acting in opposite directions, so the combined x-directed force, F_{Rx} , is smaller than either of the components, and directed in the $+x$ direction.



$$F_{Rx} := F_{x1} + (-F_{x2})$$

$$F_{Rx} = 93.302 \text{ N}$$

The minus sign was included before F_{x2} because it is directed in the $-x$ direction. The result is an x -component of the resultant force of 93 N in the $+x$ direction.

Once the x - and y -components of the resultant force have been determined, the magnitude can be calculated using

$$F_R = \sqrt{F_{Rx}^2 + F_{Ry}^2}$$

The Mathcad equation is basically the same...

$$F_R := \sqrt{F_{Rx}^2 + F_{Ry}^2}$$

$$F_R = 401.312 \text{ N}$$

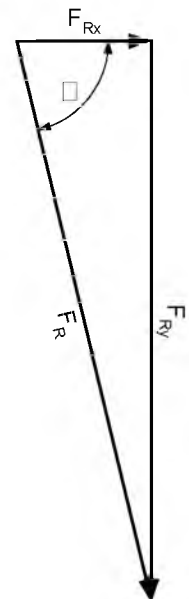
The angle of the resultant force can be calculated using any of three functions in Mathcad:

Function	Argument(s)	Notes
$\text{atan}(\text{abs}(F_x / F_y))$	one argument: $\text{abs}(F_x / F_y)$	Returns the included angle
$\text{atan2}(F_x, F_y)$	two arguments: F_x and F_y	Returns the coordinate direction angle Angle value is always between 0 and π radians (0 and 180°) A negative sign on the angle indicates a result in one of the lower quadrants of the Cartesian coordinate system
$\text{angle}(F_x, F_y)$	two arguments: F_x and F_y	Returns the positive angle from the positive x-axis to the vector Angle value always between 0 and 2π radians (0 and 360°) An angle value greater than 180° (π radians) indicates a result in one of the lower quadrants of the Cartesian coordinate system

The $\text{atan2}()$ function is used here, and F_{Ry} is negative because it is acting in the $-y$ direction.

$$\begin{aligned}
 F_{Rx} &:= 93.302 \text{ N} & F_{Ry} &:= -390.316 \text{ N} \\
 \theta &:= \text{atan2}(F_{Rx}, F_{Ry}) \\
 \theta &= 76.556 \text{ deg}
 \end{aligned}$$

The deg unit was used to display the calculated angle in degrees. If it had been left off, the result would have been displayed using Mathcad's default units, radians.



Answer, part b)

The net force (magnitude and direction) on the hook is now known:

$F_R = 401 \text{ N}$ (about 90 lbf) acting at an angle 76.6° below the x-axis.

Annotated Mathcad Worksheet

Determine the x- and y-components of the two forces (270 N at -55°, and 180 N at -110°)

Note: These trig. calculations use the included angles (55° and 20°), with minus signs added to both y-component equations to indicate that the forces act in the -y direction, and to the F_{x2} equation to show that this force acts in the -x direction.

$$F_{x1} := (270\text{N}) \cos(55^\circ)$$

$$F_{x1} = 154.866\text{N}$$

$$F_{x2} := -(180\text{N}) \sin(20^\circ)$$

$$F_{x2} = -61.564\text{N}$$

$$F_{y1} := -(270\text{N}) \sin(55^\circ)$$

$$F_{y1} = -221.171\text{N}$$

$$F_{y2} := -(180\text{N}) \cos(20^\circ)$$

$$F_{y2} = -169.145\text{N}$$

^^^ The larger boy exerts the greater force in the -y direction.

Sum the y-components of the two forces to determine the y-component of the resultant force.

$$F_{Ry} := F_{y1} + F_{y2}$$

$$F_{Ry} = -390.316\text{N}$$

Sum the x-components of the two forces to determine the x-component of the resultant force.

$$F_{Rx} := F_{x1} + F_{x2}$$

A simple addition is used here since F_{x2} is already negative.

$$F_{Rx} = 93.302\text{N}$$

Calculate the magnitude of the resultant force.

$$F_R := \sqrt{F_{Rx}^2 + F_{Ry}^2}$$

$$F_R = 401.312\text{N}$$

Calculate the angle of the resultant force (in degrees from the x-axis).

$$\theta := \text{atan2}(F_{Rx}, F_{Ry})$$

$$\theta = -76.556^\circ$$

The minus sign indicates below the x-axis.

2

Dot Products

Ref: Hibbeler § 2.9, Bedford & Fowler: Statics § 2.5

Taking the *dot product* of an arbitrary vector with a *unit vector* oriented along a coordinate direction yields the *projection* of the arbitrary vector along the coordinate axis. When this scalar (magnitude) is multiplied by a unit vector in the coordinate direction, the result is the vector component in that coordinate direction. This is one common use of the dot product. The other is finding the angle between two vectors.

The dot product (or scalar product) can be calculated in two ways:

- In trigonometric terms, the equation for a dot product is written as

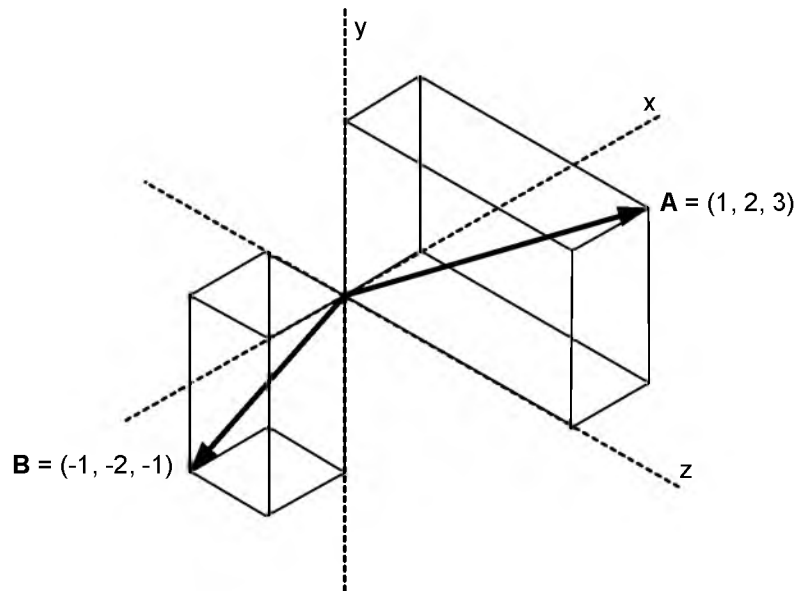
$$\mathbf{A} \cdot \mathbf{B} = A B \cos(\theta)$$

Where θ is the angle between arbitrary vectors \mathbf{A} and \mathbf{B} .

- In matrix form, the equation is written

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

Mathcad provides a dot product operator on the matrix toolbar to automatically perform the calculations required by the matrix form of the dot product. If you have two vectors written in matrix form, such as



Then $\mathbf{A} \cdot \mathbf{B}$ is the projection of \mathbf{A} onto \mathbf{B} (a magnitude, or scalar). Using Mathcad, the dot product is calculated like this (bold letters have not been used for matrix names in Mathcad):

$$\left| \begin{array}{l} \mathbf{A} := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \mathbf{B} := \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} \\ \mathbf{A} \cdot \mathbf{B} = -8 \end{array} \right. \quad \ll \text{uses the dot product operator from the Matrix toolbar}$$

To verify this result, we can do the math term by term...

$$\begin{array}{l} x := 0 \quad y := 1 \quad z := 2 \quad \ll \text{coordinate index definitions} \\ A_x B_x + A_y B_y + A_z B_z = 8 \end{array}$$

And we can use the trig. form of the dot product to find the angle between the two vectors. First, calculate the magnitude of the **A** and **B** vectors...

$$\begin{array}{l} A_{\text{mag}} := \sqrt{(A_x)^2 + (A_y)^2 + (A_z)^2} \quad A_{\text{mag}} = 3.742 \\ B_{\text{mag}} := \sqrt{(B_x)^2 + (B_y)^2 + (B_z)^2} \quad B_{\text{mag}} = 2.449 \end{array}$$

Then find the angle between the vectors using Mathcad's `acos()` function.

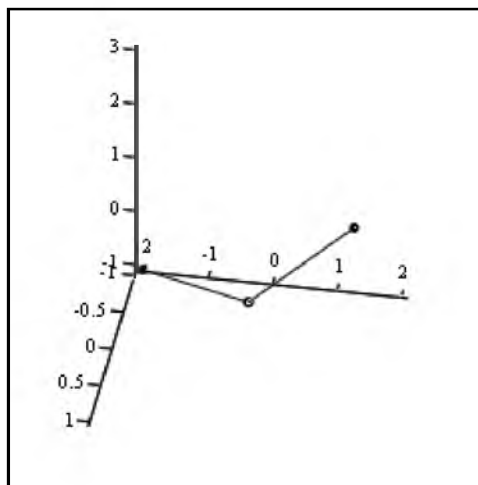
$$\phi := \arccos\left(\frac{A \cdot B}{A_{\text{mag}} B_{\text{mag}}}\right) \quad \phi = 150.794 \text{deg}$$

Finally, you can use a 3-d scatter plot in Mathcad to help you visualize these vectors, to see if your computed results make sense. To create the 3-d scatter plot, you must define the x, y, and z coordinates of the endpoints of each vector.

$$\begin{array}{l} X := \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad Y := \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix} \quad Z := \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix} \end{array}$$

Here, the top-left elements form the starting coordinates for the **A** vector: (0, 0, 0). The lower-left elements for the end coordinates for the **A** vector: (1, 2, 3). Similarly, the right columns in each matrix represent the starting (0, 0, 0) and ending (-1, -2, -1) coordinates of the **B** vector.

To create the graph, use the menu commands: Insert/Graph/3-d Scatter Plot, then enter (X, Y, Z) in the graph placeholder (the parentheses are required).

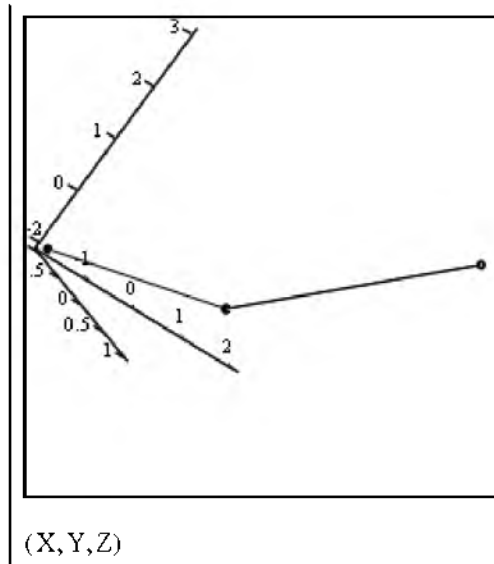


(X, Y, Z)

Note that the axes do not intercept at (0, 0, 0). Actually, the middle point on the plot is at (0, 0, 0). Also, the lines between the points are not displayed by default. To show the lines, double-click on the plot to open the 3-d Plot Format dialog, then select the Appearance tab, and select the Lines option.

The usefulness of the 3-d plot for visualization is the ability to rotate the graph to see how the plot looks from different angles. To rotate the plot, simply click inside the plot area, hold the left mouse button down, and drag the mouse pointer around the plot. The curve will respond to the location of the pointer.

In the graph below, the plot has been rotated until the plane of the vectors is approximately parallel with the page. In this view, the calculated angle of 150° looks to be about right for these vectors.



Annotated Mathcad Worksheet

Define the vectors

$$A := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad B := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Take the dot product

$$A \cdot B = 8 \quad \ll \text{uses the dot product operator from the Matrix toolbar}$$

Check Mathcad's dot product operator by calculating the dot product explicitly...

$$x := 0 \quad y := 1 \quad z := 2 \quad \ll \text{coordinate index definitions (could use 0, 1, 2 directly)}$$

$$A_x B_x + A_y B_y + A_z B_z = 8$$

To calculate the angle between the vectors, first need the magnitude of each vector.

$$A_{\text{mag}} := \sqrt{(A_x)^2 + (A_y)^2 + (A_z)^2} \quad A_{\text{mag}} = 3.742$$

$$B_{\text{mag}} := \sqrt{(B_x)^2 + (B_y)^2 + (B_z)^2} \quad B_{\text{mag}} = 2.449$$

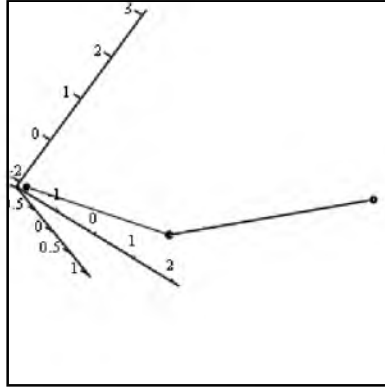
Then calculate the angle...

$$\theta := \arccos\left(\frac{A \cdot B}{A_{\text{mag}} B_{\text{mag}}}\right) \quad \theta = 150.794 \text{deg}$$

To visualize the vectors, create matrices describing the starting and ending coordinates of each vector.

$$X := \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad Y := \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix} \quad Z := \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix}$$

Then insert a 3-d Scatter Plot



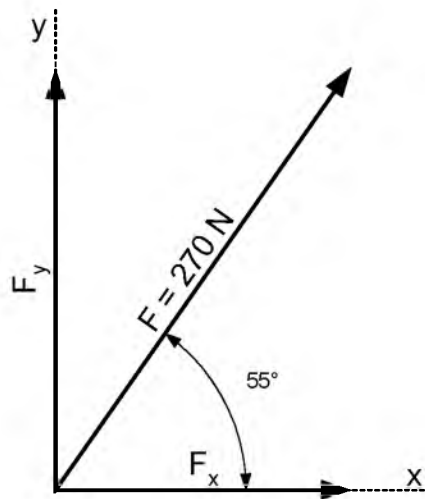
Click and drag inside the plot area to rotate the graph.

(X,Y,Z)

Example: Finding the Components of a Force Vector

A force acts on a fixed pin with a magnitude of 270 N (about 60 lb_f) at an angle of 55° from horizontal.

- Find the horizontal and vertical components of the force using dot products.



Solution

To find the magnitude of the x-component of \mathbf{F} , calculate the dot product of \mathbf{F} and a unit vector in the x direction, \mathbf{u}_x .

$$F_x = F \cos(\theta) = \frac{\mathbf{F} \cdot \mathbf{u}_x}{u_x}$$

Note: Most texts do not show u_x in the denominator of the last term of this series of equalities, since the magnitude of a unit vector is known to be one. But, if you plan to use units in Mathcad, the u_x in the denominator is needed to make the units work out.

$$\begin{aligned} F &:= 270 \text{ N} & \theta &:= 55^\circ \\ u_x &:= 1 \text{ N} & & \text{(a unit vector has a magnitude of 1, units here are Newtons)} \\ F_x &:= F \cos(\theta) \\ F_x &= 154.866 \text{ N} \end{aligned}$$

Similarly, the vertical component of **F** is calculated as:

$$\begin{aligned} F_y &:= F \cos[(90 - 55)^\circ] \\ F_y &= 221.171 \text{ N} \end{aligned}$$

Alternatively, the **F** and u_x vectors can be written in vector form.

$$\begin{aligned} F &:= \begin{bmatrix} 270 \cos(55^\circ) \\ 270 \cos[(90 - 55)^\circ] \\ 0 \end{bmatrix} \text{ N} \\ u_x &:= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ N} \end{aligned}$$

And the dot product can be used to calculate the horizontal component of **F**.

$$\begin{aligned} u_{x_mag} &:= 1 \text{ N} \\ F_x &:= \frac{F \cdot u_x}{u_{x_mag}} \\ F_x &= 154.866 \text{ N} \end{aligned}$$

Similarly, we can define a unit vector in the y-direction,

$$u_y := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ N} \quad u_{y_mag} := 1 \text{ N}$$

And calculate the vertical component of the force.

$$\begin{aligned} F_y &:= \frac{F \cdot u_y}{u_{y_mag}} \\ F_y &= 221.171 \text{ N} \end{aligned}$$

3

Equilibrium of a Particle, Free-Body Diagrams

Ref: Hibbeler § 3.3, Bedford & Fowler: Statics § 3.2-3.3

When a body is either not moving (zero velocity), or moving at a constant velocity (speed and direction), the sum of the external forces on the body is zero, and the body is said to be in *equilibrium*.

$$\sum \mathbf{F} = 0$$

or, for two-dimensional equilibrium,

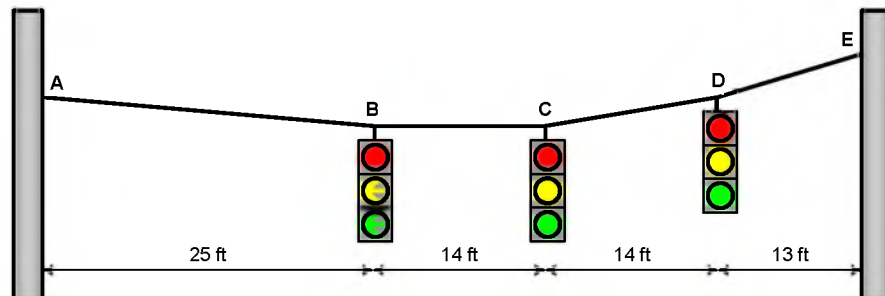
$$\sum F_x = 0$$

$$\sum F_y = 0$$

The picture showing the external forces acting on the object is called a *free-body diagram*. A free-body diagram is used to help solve problems both in statics and dynamics.

Example: Forces on Traffic Light Suspension Cables

Three traffic lights have been suspended between two poles at an intersection, as shown below.



Each of the three lights has a mass of 18 kg (approx. 40 lb). (Assume zero mass cables.) The tension in the cables has been adjusted such that the lights at points B and C are at the same height, and cable section AB is at an angle of 5° from horizontal.

- Determine the downward force at points B, C, and D due to the mass of the lights.
- Draw a free-body diagram at point B, and use it to find the vertical and horizontal components of force in cable AB.
- Repeat part b. for the other lights, determining the force components in each cable section, and the angle of each cable section (measured from the +x direction).

Solution: Part a.

The mass of each traffic light is being acted on by gravity, so the force is calculated as

$$F_y = m g$$

In Mathcad, this calculation looks like this:

$$m := 18 \text{ kg}$$

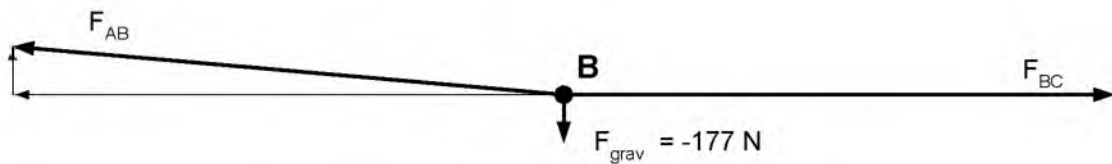
$$F_{\text{grav}} := -m \cdot g \quad \ll 'g' \text{ is predefined as } 9.8 \text{ m/s}^2 \text{ in Mathcad}$$

$$F_{\text{grav}} = -177 \text{ N}$$

So the downward force exerted by each traffic light is 177 N. The minus sign has been included to show that the force is downward, i.e., in the $-y$ direction.

If the lights are not moving up or down, the cable must apply an equal but oppositely directed force on the light, since the vertical force components must sum to zero if the body is in equilibrium.

Part b. – Free-Body Diagram for Light at B



Since cable section BC is horizontal, there is no vertical component of force in cable section BC. So, all of the weight of the traffic light at B must be carried by cable AB, more specifically, by the vertical component of force in cable AB.

$$F_{AB_y} := -F_{\text{grav}}$$

$$F_{AB_y} = 177 \text{ N}$$

The horizontal component of force in cable AB can be determined using the specified angle (cable AB is 5° from horizontal, or 175° from $+x$).

$$F_{AB_x} := \frac{F_{AB_y}}{\tan(175^\circ)}$$

$$F_{AB_x} = -2018 \text{ N}$$

The horizontal component is 2018 N (about 450 lb_f) acting in the $-x$ direction.

The force acting in the direction of the cable has a magnitude of 2025 N.

$$F_{AB} := \sqrt{F_{AB_y}^2 + F_{AB_x}^2}$$

$$F_{AB} = 2025 \text{ N}$$

Finally, if the light at point B is not moving to the left or right, the sum of the horizontal forces acting on point B must be zero, so the horizontal component of force in section BC is +2018 N. Since there is no vertical component of force in section BC, this is also the total force on section BC at point B.

$$F_{BC_x} := -F_{AB_x} \quad \ll \text{ since the sum of the x-components of force is zero}$$

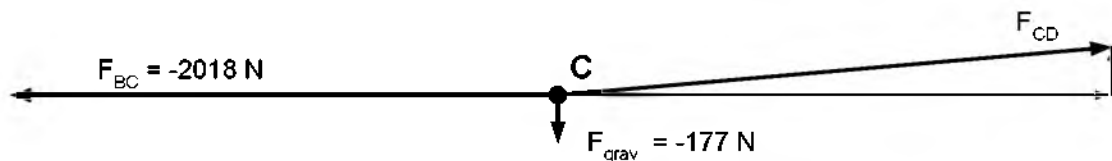
$$F_{BC_x} = 2018 \text{ N}$$

$$F_{BC_y} := 0 \text{ N}$$

$$F_{BC} := \sqrt{F_{BC_y}^2 + F_{BC_x}^2}$$

$$F_{BC} = 2018 \text{ N}$$

Part b. – Free-Body Diagram for Light at C



The free body diagram for the light at point C is constructed using the following concepts:

- ☐ If the light at point C is not moving left or right, then the horizontal component of force F_{CD} must be $+2018 \text{ N}$.
- ☐ If the light at point C is not moving up or down, then the vertical component of force F_{CD} must be $+177 \text{ N}$.

This is essentially the same as the free-body diagram at point B, just flipped left to right. So the magnitude of force F_{CD} is 2025 N , and acts at 5° from horizontal. This can be verified as follows:

$$F_{CD_x} := -F_{BC}$$

$$F_{CD_x} = 2018 \text{ N}$$

$$F_{CD_y} := -F_{\text{grav}}$$

$$F_{CD_y} = 177 \text{ N}$$

$$F_{CD} := \sqrt{F_{CD_y}^2 + F_{CD_x}^2}$$

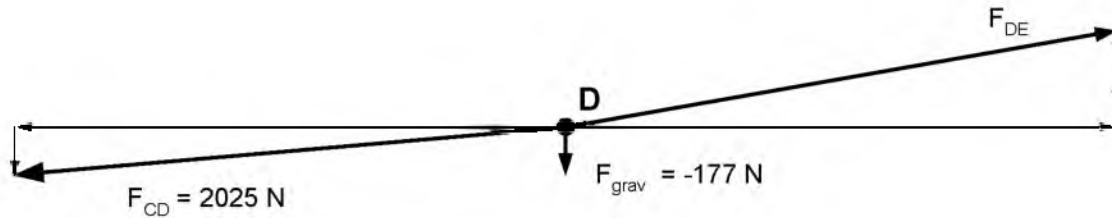
$$F_{CD} = 2025 \text{ N}$$

$$\theta_{CD} := \text{atan}\left(\frac{F_{CD_y}}{F_{CD_x}}\right)$$

$$\theta_{CD} = 5^\circ$$

Note: The value of F_{BC} used in this section of the Mathcad worksheet is -2018 N , as shown in the free-body diagram for point C. See the annotated Mathcad Worksheet to see how this sign change is handled for the complete problem solution.

Part b. – Free-Body Diagram for Light at D



The weight of the traffic light at point D exerts a downward force, F_{grav} , of 177 N. In addition, force F_{CD} , acting on point D, has a vertical component of 177 N, also acting downward. If the light at point D is not moving up or down, then these downward forces must be counterbalanced by the vertical component of force F_{DE} .

$$F_{DE_y} := (F_{grav} + F_{CD_y})$$

$$F_{DE_y} = 353 \text{ N}$$

The horizontal component of force F_{DE} must be equal to $-F_{CD_x}$ if the light at point D is to be stationary.

$$F_{DE_x} := F_{CD_x}$$

$$F_{DE_x} = 2018 \text{ N}$$

Once the horizontal and vertical components are known, the angle of cable DE can be determined.

$$\theta_{DE} := \text{atan}\left(\frac{F_{DE_y}}{F_{DE_x}}\right)$$

$$\theta_{DE} = 9.925 \text{ deg}$$

Annotated Mathcad Solution

Traffic Light Suspension Cable

Part a.

$$m := 18 \text{ kg}$$

$$F_{\text{grav}} := m \cdot g \quad \ll 'g' \text{ is predefined as } 1.8 \text{ m/s}^2 \text{ in Mathcad}$$

$$F_{\text{grav}} = 177 \text{ N}$$

Part b. -- Light at Point B

$$F_{AB_y} := F_{\text{grav}}$$

$$F_{AB_y} = 177 \text{ N}$$

$$F_{AB_x} := \frac{F_{AB_y}}{\tan(17.5 \text{ deg})}$$

$$F_{AB_x} = 2018 \text{ N} \quad F_{AB_x} = 454 \text{ lbf}$$

$$F_{BC_x} := F_{AB_x} \quad \ll \text{since the sum of the x-components of force is zero}$$

$$F_{BC_x} = 2018 \text{ N}$$

$$F_{BC_y} := 0 \text{ N}$$

$$F_{BC} := \sqrt{F_{BC_y}^2 + F_{BC_x}^2}$$

$$F_{BC} = 2018 \text{ N} \quad \ll F_{BC} \text{ acting on point B is in the +x direction}$$

$$F_{AB} := \sqrt{F_{AB_y}^2 + F_{AB_x}^2}$$

$$F_{AB} = 2025 \text{ N} \quad F_{AB} = 455 \text{ lbf}$$

Part b. -- Light at Point C

$$F_{BC} := F_{BC} \quad \ll F_{BC} \text{ acting on point C is in the -x direction}$$

$$F_{CD_x} := F_{BC}$$

$$F_{CD_x} = 2018 \text{ N}$$

$$F_{CD_y} := F_{\text{grav}}$$

$$F_{CD_y} = 177 \text{ N}$$

$$F_{CD} := \sqrt{F_{CD_y}^2 + F_{CD_x}^2}$$

$$F_{CD} = 2025 \text{ N} \quad F_{CD} = 455 \text{ lbf}$$

$$\theta_{CD} := \text{atan}\left(\frac{F_{CD_y}}{F_{CD_x}}\right)$$

$$\theta_{CD} = 5 \text{ deg}$$

$$F_{CD_y} := -F_{CD_y}$$

<< These force components, *acting on point D* , are in the opposite direction of the same components, *acting on point C* .

$$F_{CD_x} := -F_{CD_x}$$

$$F_{DE_y} := -(F_{grav} + F_{CD_y})$$

$$F_{DE_y} = 353 \text{ N}$$

$$F_{DE_x} := -F_{CD_x}$$

$$F_{DE_x} = 2018 \text{ N}$$

$$\theta_{DE} := \text{atan}\left(\frac{F_{DE_y}}{F_{DE_x}}\right)$$

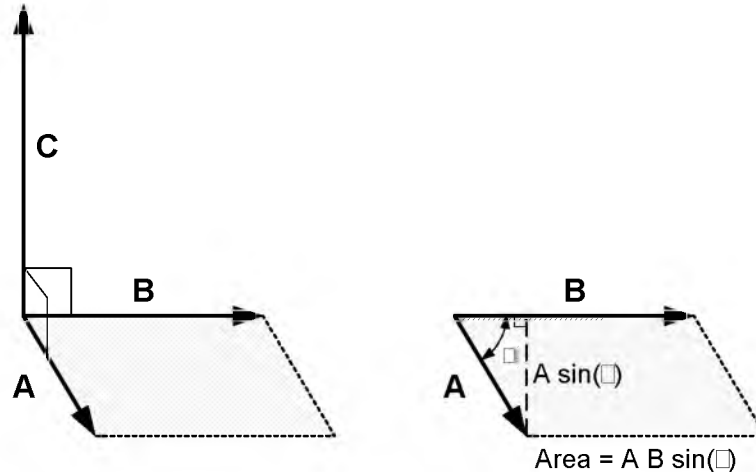
$$\theta_{DE} = 9.925 \text{ deg}$$

4

Cross Products and Moments of Force

Ref: Hibbeler § 4.2-4.3, Bedford & Fowler: Statics § 2.6, 4.3

In geometric terms, the *cross product* of two vectors, **A** and **B**, produces a new vector, **C**, with a direction perpendicular to the plane formed by **A** and **B** (according to right-hand rule) and a magnitude equal to the area of the parallelogram formed using **A** and **B** as adjacent sides.



The cross product is used to find the moment of force. An example of this will be shown after describing the basic mathematics of the cross product operation.

The cross product (or vector product) can be calculated in two ways:

- In trigonometric terms, the equation for a dot product is written as

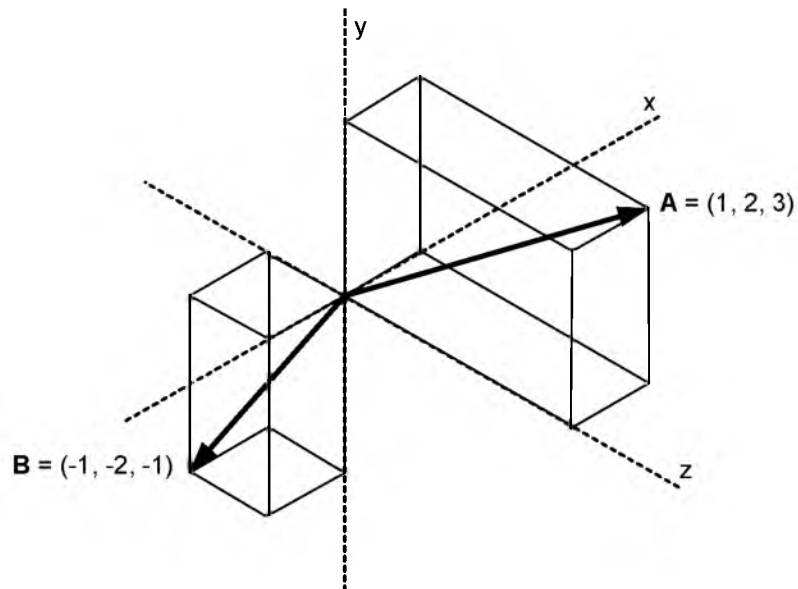
$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = A B \sin(\theta) \mathbf{u}_C$$

Where θ is the angle between arbitrary vectors **A** and **B**, and \mathbf{u}_C is a unit vector in the direction of **C** (perpendicular to **A** and **B**, using right-hand rule).

- In matrix form, the equation is written in using components of vectors **A** and **B**, or as a determinant. Symbols **i**, **j**, and **k** represent unit vectors in the coordinate directions.

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (A_y B_z - A_z B_y) \mathbf{i} - (A_x B_z - A_z B_x) \mathbf{j} + (A_x B_y - A_y B_x) \mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \end{aligned}$$

Mathcad provides a cross product operator on the matrix toolbar to automatically perform the calculations required by the matrix form of the dot product. If you have two vectors written in matrix form, such as



Then $\mathbf{A} \cdot \mathbf{B}$ can be calculated like this (bold letters have not been used for matrix names in Mathcad):

$$\left| \begin{array}{l} \mathbf{A} := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \mathbf{B} := \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} \\ \mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} \quad \ll \text{uses the cross product operator from the Matrix toolbar} \end{array} \right|$$

To verify this result, we can do the math term by term...

$$\left| \begin{array}{l} x := 0 \quad y := 1 \quad z := 2 \quad \ll \text{coordinate index definitions} \\ \begin{bmatrix} A_y B_z - A_z B_y \\ A_x B_z - A_z B_x \\ A_x B_y - A_y B_x \end{bmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} \end{array} \right|$$

Or, we can use the trig. form of the cross product. First, we calculate the magnitude of the \mathbf{A} and \mathbf{B} vectors using the determinant operator from the Matrix toolbar...

$$\left| \begin{array}{l} A_{\text{mag}} := |\mathbf{A}| \quad A_{\text{mag}} = 3.742 \\ B_{\text{mag}} := |\mathbf{B}| \quad B_{\text{mag}} = 2.449 \end{array} \right|$$

Then find the angle between vectors \mathbf{A} and \mathbf{B} using Mathcad's $\text{acos}()$ function.

$$\left| \begin{array}{l} \phi := \text{acos} \left(\frac{\mathbf{A} \cdot \mathbf{B}}{A_{\text{mag}} B_{\text{mag}}} \right) \quad \phi = 150.794 \text{deg} \end{array} \right|$$

The magnitude of the \mathbf{C} matrix can then be calculated...

$$C_{\text{mag}} := A_{\text{mag}} B_{\text{mag}} \sin(\alpha) \quad C_{\text{mag}} = 4.472$$

The direction of **C** is perpendicular to the plane formed by **A** and **B**, and is found using the cross product. To obtain the direction cosines of **C**, divide the cross product of **A** and **B** by its magnitude.

$$\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A} \cdot \mathbf{B}|} = \begin{pmatrix} 0.894 \\ -0.447 \\ 0 \end{pmatrix}$$

$$\alpha := \arccos(0.894) \quad \alpha = 26.62^\circ \quad \ll \text{from } +x$$

$$\beta := \arccos(-0.447) \quad \beta = 116.551^\circ \quad \ll \text{from } +y$$

$$\gamma := \arccos(0) \quad \gamma = 90^\circ \quad \ll \text{from } +z$$

This is, of course, equivalent to

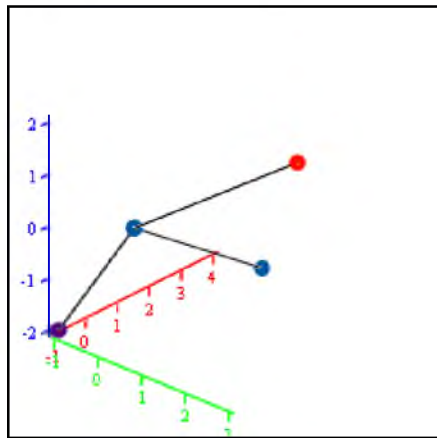
$$\mathbf{C} := \mathbf{A} \cdot \mathbf{B} \quad \frac{\mathbf{C}}{C_{\text{mag}}} = \begin{pmatrix} 0.894 \\ -0.447 \\ 0 \end{pmatrix}$$

The vectors can be graphed to see how the cross product works. The plot on the left shows the original plot, with the axes oriented in the same way as the drawing on the second page. In the plot on the right the axes have been rotated to show that vector **C** is perpendicular to the plane formed by vectors **A** and **B**.

$$x := \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 4 \end{pmatrix}$$

$$y := \begin{pmatrix} 0 & 0 & 0 \\ 2 & -2 & -2 \end{pmatrix}$$

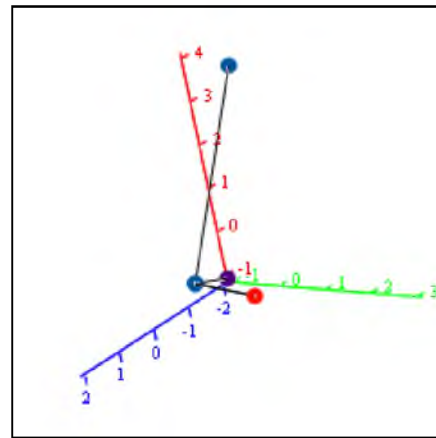
$$z := \begin{pmatrix} 0 & 0 & 0 \\ 3 & -1 & 0 \end{pmatrix}$$



(x,y,z)

x - red
y - blue
z - green

A - red
B - violet
C - blue



(x,y,z)

Annotated Mathcad Worksheet

Define the vectors

$$A := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad B := \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Take the cross product

$$A \cdot B = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \quad \ll \text{uses the cross product operator from the Matrix toolbar}$$

Check Mathcad's cross product operator by calculating the cross product explicitly...

$$x := 0 \quad y := 1 \quad z := 2 \quad \ll \text{coordinate index definitions}$$

$$\begin{bmatrix} A_y B_z - A_z B_y \\ A_x B_z - A_z B_x \\ A_x B_y - A_y B_x \end{bmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

Use the trigonometric form of the cross product to find the magnitude of the **C** vector.

First, find the magnitude of the **A** and **B** vectors using the determinant operator from the Matrix toolbar.

$$A_{\text{mag}} := |A| \quad A_{\text{mag}} = 3.742$$

$$B_{\text{mag}} := |B| \quad B_{\text{mag}} = 2.449$$

Then, find the angle between vectors **A** and **B**.

$$\theta := \arccos \left(\frac{A \cdot B}{A_{\text{mag}} B_{\text{mag}}} \right) \quad \theta = 150.794 \text{deg}$$

Finally, solve for the magnitude of the **C** vector.

$$C_{\text{mag}} := A_{\text{mag}} B_{\text{mag}} \sin(\theta) \quad C_{\text{mag}} = 4.472$$

Solve for the direction cosines of the **C** vector.

$$\frac{A \cdot B}{|A \cdot B|} = \begin{pmatrix} 0.894 \\ 0.447 \\ 0 \end{pmatrix} \quad \text{- or -} \quad C := A \cdot B \quad \frac{C}{C_{\text{mag}}} = \begin{pmatrix} 0.894 \\ 0.447 \\ 0 \end{pmatrix}$$

$$\theta_x := \arccos(0.894) \quad \theta_x = 26.62 \text{deg} \quad \ll \text{from +x}$$

$$\theta_y := \arccos(0.447) \quad \theta_y = 116.551 \text{deg} \quad \ll \text{from +y}$$

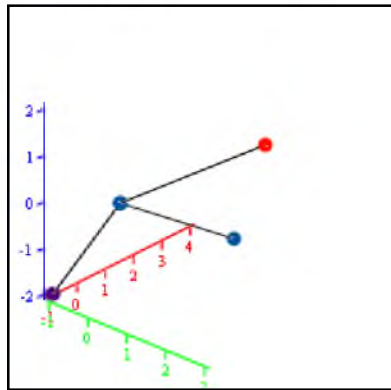
$$\theta_z := \arccos(0) \quad \theta_z = 90 \text{deg} \quad \ll \text{from +z}$$

Plot the **A**, **B**, and **C** vectors

$$x := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 4 \end{pmatrix}$$

$$y := \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \end{pmatrix}$$

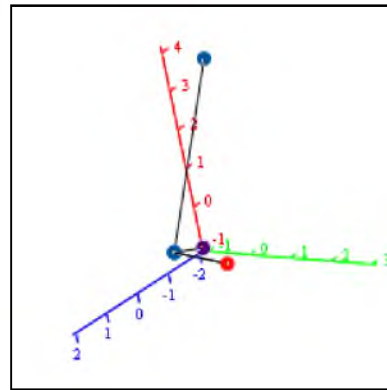
$$z := \begin{pmatrix} 0 & 0 & 0 \\ 3 & 1 & 0 \end{pmatrix}$$



(x, y, z)

x - red
y - blue
z - green

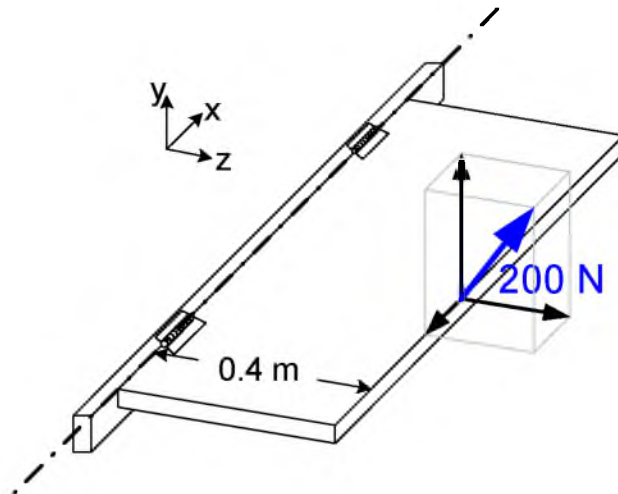
A - red
B - violet
C - blue



(x, y, z)

Example: Find the Moment of a Force on a Line

A force of $F = 200$ N acts on the edge of a hinged shelf, 0.40 m from the pivot point.



The 200 N force has the following components:

$$F_x = -40 \text{ N}$$

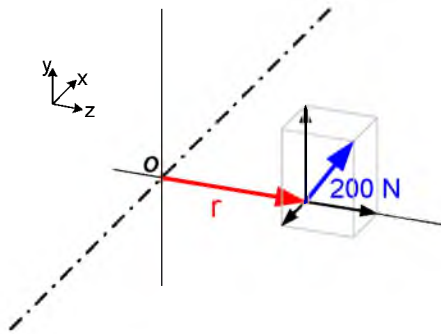
$$F_y = 157 \text{ N}$$

$$F_z = 118 \text{ N}$$

Only the y -component of F will tend to cause rotation on the hinges. What is the moment of the force about the line passing through the hinges (the x axis)?

Solution Using the Cross Product

We begin by defining the vector r which starts at the x axis (the line through the hinges) and connects to the point at which force F acts.



Since the shelf was 0.40 m wide, \mathbf{r} has a magnitude of 0.40, is oriented in the +z direction, and can be written in component form as

$$\mathbf{r} := \begin{pmatrix} 0 \\ 0 \\ 0.4 \end{pmatrix} \quad \mathbf{F} := \begin{pmatrix} -40 \\ 157 \\ 118 \end{pmatrix}$$

The moment of force \mathbf{F} about the point O is found using the cross product of \mathbf{r} with \mathbf{F} .

$$\mathbf{M}_O := \mathbf{r} \times \mathbf{F} \quad \mathbf{M}_O = \begin{pmatrix} -62.8 \\ 16 \\ 0 \end{pmatrix}$$

$$M_{\text{Mag}} := |\mathbf{M}_O| \quad M_{\text{Mag}} = 64.806$$

However, the moment of force \mathbf{F} about the x axis requires an additional dot product with a unit vector in the x-direction, and is found as

$$\mathbf{u}_x := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{r} := \begin{pmatrix} 0 \\ 0 \\ 0.4 \end{pmatrix} \quad \mathbf{F} := \begin{pmatrix} -40 \\ 157 \\ 118 \end{pmatrix}$$

$$M_L := \mathbf{u}_x \cdot (\mathbf{r} \times \mathbf{F}) \quad M_L = -62.8$$

The minus sign indicates that the moment is directed in the $-x$ direction.

Solution Using Scalars

The moment of force \mathbf{F} about the x axis can also be determined by multiplying the y-component of \mathbf{F} and the perpendicular distance between the point at which \mathbf{F} acts and the x axis.

$$M_L = F_y d$$

Since the component of \mathbf{F} in the y-direction is known (157 N), and the perpendicular distance is 0.4 m, the moment can be calculated from these quantities.

$$F_y := 157 \quad d := 0.4$$

$$M_L := F_y d \quad M_L = 62.8$$

The direction must be determined using the right-hand rule, where the thumb indicates the direction when the fingers are curled around the x axis in the direction of the rotation caused by F_y .

Annotated Worksheet

Define the vectors

$$\mathbf{r} := \begin{pmatrix} 0 \\ 0 \\ 0.4 \end{pmatrix} \quad \mathbf{F} := \begin{pmatrix} -40 \\ 157 \\ 118 \end{pmatrix}$$

Take the cross product of \mathbf{r} with \mathbf{F} to get the moment about point O (the origin).

$$\mathbf{M}_O := \mathbf{r} \times \mathbf{F} \quad \mathbf{M}_O = \begin{pmatrix} -62.8 \\ 16 \\ 0 \end{pmatrix}$$

Note: This is not the solution to the stated question. The question asked for the moment about the x axis. That will be calculated next.

$$M_{\text{Mag}} := |\mathbf{M}_O| \quad M_{\text{Mag}} = 64.806$$

Declare a unit vector in the x-direction in order to calculate the moment about the x axis.

$$\mathbf{u}_x := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{r} := \begin{pmatrix} 0 \\ 0 \\ 0.4 \end{pmatrix} \quad \mathbf{F} := \begin{pmatrix} -40 \\ 157 \\ 118 \end{pmatrix}$$

Calculate the moment about the x axis

$$M_L := \mathbf{u}_x \cdot (\mathbf{r} \times \mathbf{F}) \quad M_L = -62.8$$

Check the result using scalar math

$$F_y := 157 \quad d := 0.4$$

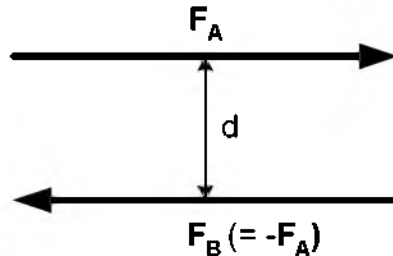
$$M_L := F_y \cdot d \quad M_L = 62.8$$

5

Moment of a Couple

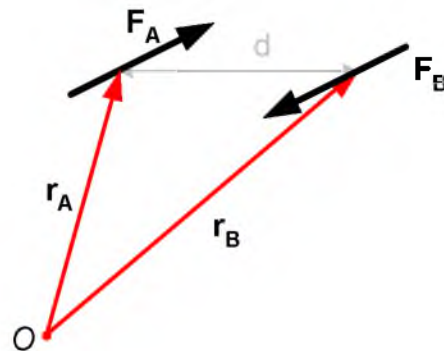
Ref: Hibbeler § 4.6, Bedford & Fowler: Statics § 4.4

A *couple* is a pair of forces, equal in magnitude, oppositely directed, and displaced by perpendicular distance, d .



Since the forces are equal and oppositely directed, the resultant force is zero. But the displacement of the force couple (d) does create a *couple moment*.

The moment, M , about some arbitrary point O can be calculated.



$$\begin{aligned} \mathbf{M} &= \mathbf{r}_A \cdot \mathbf{F}_A + \mathbf{r}_B \cdot \mathbf{F}_B \\ &= \mathbf{r}_A \cdot \mathbf{F}_A + \mathbf{r}_B \cdot (-\mathbf{F}_A) \end{aligned}$$

If point O is placed on the line of action of one of the forces, say F_B , then that force causes no rotation (or tendency toward rotation) and the calculation of the moment is simplified.

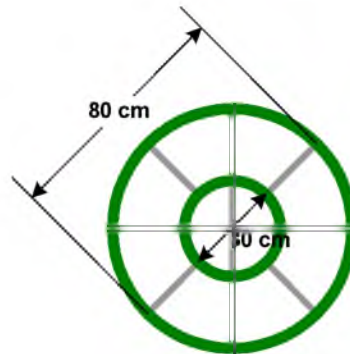


$$\mathbf{M} = \mathbf{r} \cdot \mathbf{F}_A$$

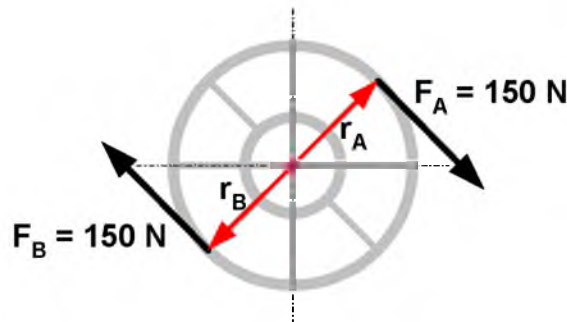
This is a significant result: The couple moment, M , depends only on the position vector r between forces F_A and F_B . The couple moment does not have to be determined relative to the location of a point or an axis.

Example A: Moment from a Large Hand Wheel

The stem on a valve has two hand wheels: a small wheel (30 cm diameter) used to spin the valve quickly as it is opened and closed, and a large wheel (80 cm diameter) that may be used to free a stuck valve, or seat the valve tightly when it is fully closed.



If the operator can impose a force of 150 N on each side of the large wheel (a force couple), what moment is imposed on the valve stem?



Solution 1

As drawn, both the force and position vectors have x- and y-components. The vectors may be defined as:

$$\begin{aligned} \mathbf{F}_A &:= \begin{pmatrix} 106.06 \\ -106.06 \\ 0 \end{pmatrix} \text{ N} & \mathbf{F}_B &:= \begin{pmatrix} -106.06 \\ 106.06 \\ 0 \end{pmatrix} \text{ N} \\ \mathbf{r}_A &:= \begin{pmatrix} 28.284 \\ 28.284 \\ 0 \end{pmatrix} \text{ cm} & \mathbf{r}_B &:= \begin{pmatrix} -28.284 \\ 28.284 \\ 0 \end{pmatrix} \text{ cm} \end{aligned}$$

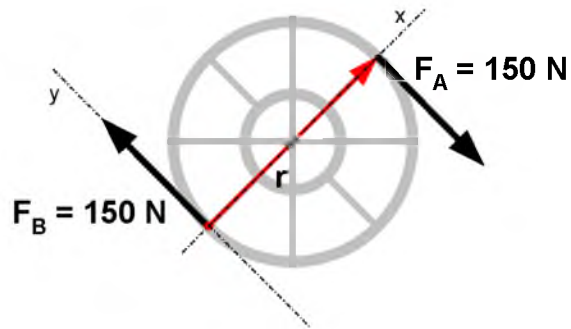
The moment of the couple can be calculated using the cross product operator on the Matrix toolbar.

$$\begin{aligned} \mathbf{M} &:= \mathbf{r}_A \cdot \mathbf{F}_A + \mathbf{r}_B \cdot \mathbf{F}_B & \mathbf{M} &= \begin{pmatrix} 0 \\ 0 \\ -120 \end{pmatrix} \text{ Nm} \end{aligned}$$

The result is a couple moment of 120 N·m directed in the $-z$ direction (into the page).

Solution 2

Perhaps a more reasonable positioning of the axes for this problem might look like this:



In this case, the vector definitions become:

$$\left| \begin{array}{l} \mathbf{F}_A := \begin{pmatrix} 0 \\ -150 \\ 0 \end{pmatrix} \text{ N} \quad \mathbf{r} := \begin{pmatrix} 80 \\ 0 \\ 0 \end{pmatrix} \text{ cm} \end{array} \right|$$

Since the position vector \mathbf{r} originates from the line of action of force \mathbf{F}_B , \mathbf{F}_B does not contribute to the moment. The moment is then calculated as

$$\left| \begin{array}{l} \mathbf{M} := \mathbf{r} \cdot \mathbf{F}_A \quad \mathbf{M} = \begin{pmatrix} 0 \\ 0 \\ -120 \end{pmatrix} \text{ N}\cdot\text{m} \end{array} \right|$$

The result is, of course, the same no matter how the axes are situated.

Solution 3: Using Scalars

Finally, the problem can also be solved using a scalar formulation. The perpendicular distance between the forces is 80 cm. With axes established as in Solution 2, the moment can be calculated as

$$\left| \begin{array}{l} \mathbf{M} := (80 \text{ cm}) \cdot (150 \text{ N}) \quad \mathbf{M} = 120 \text{ N}\cdot\text{m} \end{array} \right|$$

The direction must be determined using the right-hand rule.

Annotated Mathcad Worksheet

Solution 1

Define the vectors

$$\mathbf{F}_A := \begin{pmatrix} 106.06 \\ -106.06 \\ 0 \end{pmatrix} \text{ N} \quad \mathbf{F}_B := \begin{pmatrix} -106.06 \\ 106.06 \\ 0 \end{pmatrix} \text{ N}$$

$$\mathbf{r}_A := \begin{pmatrix} 28.284 \\ 28.284 \\ 0 \end{pmatrix} \text{ cm} \quad \mathbf{r}_B := \begin{pmatrix} -28.284 \\ 28.284 \\ 0 \end{pmatrix} \text{ cm}$$

Compute the moment using the cross product operator from the Matrix toolbar

$$\mathbf{M} := \mathbf{r}_A \cdot \mathbf{F}_A + \mathbf{r}_B \cdot \mathbf{F}_B \quad \mathbf{M} = \begin{pmatrix} 0 \\ 0 \\ -120 \end{pmatrix} \text{ N}\cdot\text{m}$$

Solution 2

Define the vectors

$$\mathbf{F}_A := \begin{pmatrix} 0 \\ -150 \\ 0 \end{pmatrix} \text{ N} \quad \mathbf{r} := \begin{pmatrix} 80 \\ 0 \\ 0 \end{pmatrix} \text{ cm}$$

Compute the moment

$$\mathbf{M} := \mathbf{r} \cdot \mathbf{F}_A \quad \mathbf{M} = \begin{pmatrix} 0 \\ 0 \\ -120 \end{pmatrix} \text{ N}\cdot\text{m}$$

Solution 3

Define the applied force and the perpendicular distance between the forces

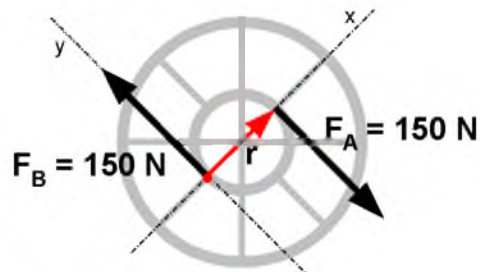
$$F := 150 \text{ N} \quad d := 80 \text{ cm}$$

Compute the moment

$$M := d \cdot F \quad M = 120 \text{ N}\cdot\text{m}$$

Example B: Moment from the Small Hand Wheel

The moment resulting from applying the same forces to the smaller hand wheel can be determined using any of the three solution procedures outlined above. Solution 2 is shown here.



Solution 2: Small Hand Wheel

Define the vectors

$$\mathbf{F}_A := \begin{pmatrix} 0 \\ -150 \\ 0 \end{pmatrix} \text{ N} \quad \mathbf{r} := \begin{pmatrix} 30 \\ 0 \\ 0 \end{pmatrix} \text{ cm}$$

Compute the moment

$$\mathbf{M} := \mathbf{r} \cdot \mathbf{F}_A \quad \mathbf{M} = \begin{pmatrix} 0 \\ 0 \\ -45 \end{pmatrix} \text{ Nm}$$

6

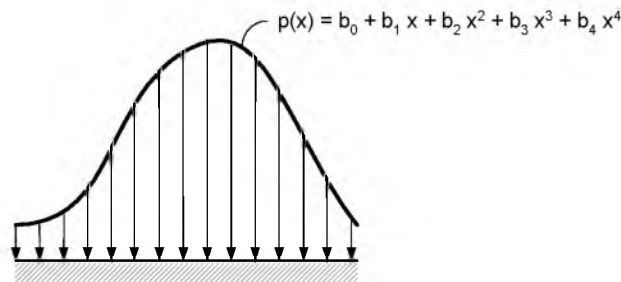
Reduction of a Simple Distributed Loading

Ref: Hibbeler § 4.10, Bedford & Fowler: Statics § 7.3

Loads are often not applied to specific points, but are *distributed* across a region. Design calculations can be simplified by finding a single equivalent force acting at a point; this is called *reduction of a distributed loading*. For loads distributed across a single direction (beam loading, for example) we need to find the both the magnitude of the equivalent force, and the position at which the equivalent force acts.

Example 1: Load Distribution Expressed as a Function of Position, x

The load on a beam is distributed as shown below:



where b_0 through b_4 are coefficients obtained by fitting a polynomial to data values (see Example 2). The x values range from 0.05 to 0.85 meters. The pressure is expressed in Pa. The values of the coefficients are tabulated below.

	$b \times 10^{-4}$
b_0	0.107
b_1	-1.68
b_2	11.9
b_3	-19.9
b_4	9.59

The data represent the pressure on the floor beneath a pile of 12-foot 2x4's, so $y = 12$ feet or 3.66 m.

Determine:

- the two-dimensional loading function, $w(x)$
- the magnitude and position of the equivalent force.

Solution 1

The two-dimensional loading function is obtained from the pressure function using the length of the boards, $y = 3.66$ m.

$$w(x) = p(x) y$$

$$= (b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4)(3.66)$$

The units on w are Pa m.

The magnitude of the resultant force is obtained by integration.

$$F_R = \int_L w(x) dx$$

Mathcad can perform this integration.

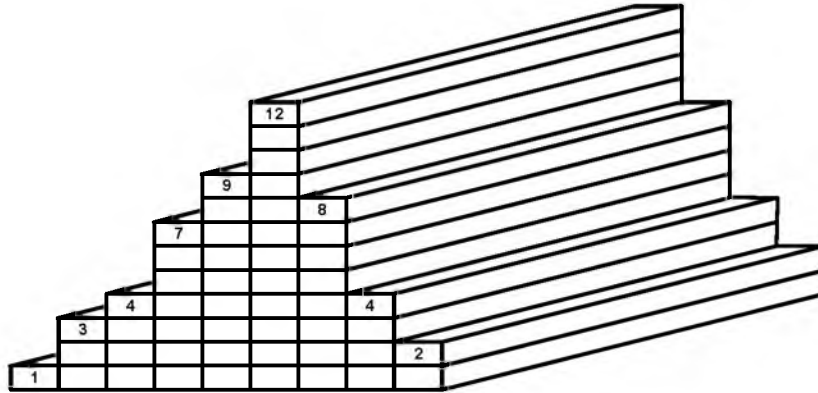
$$\begin{aligned}
 y &:= 3.66 && \text{meters} \\
 b &:= \begin{pmatrix} 1.07 \times 10^3 \\ -1.68 \times 10^4 \\ 1.19 \times 10^5 \\ -1.99 \times 10^5 \\ 9.59 \times 10^4 \end{pmatrix} \\
 p(x) &:= \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 \right) && \text{Pascals} \\
 w(x) &:= p(x) \cdot y && \text{N/m} \\
 F_R &:= \int_{0.05}^{0.85} w(x) dx \\
 F_R &= 6237 && \text{Newtons}
 \end{aligned}$$

The location at which the resultant force acts is found by calculating the centroid of the area defined by $w(x)$.

$$\begin{aligned}
 x_{loc} &:= \frac{\int_{0.05}^{0.85} x \cdot w(x) dx}{\int_{0.05}^{0.85} w(x) dx} \\
 x_{loc} &= 0.496 && \text{meters}
 \end{aligned}$$

Example 2: Tabulated Load Distribution

The function used in Example 1 was obtained by polynomial regression of a set of data values, representing the pressure acting on the floor beneath a pile of 2x4 (inch, or 50x100 mm) lumber.



Using the density of wood (approximately 700 kg/m^3) and the lumber dimensions, the pressure exerted by the wood on the floor at the bottom of each stack of lumber was calculated (see table). The reported x value represents the position of the center of each column of boards.

$x \text{ (m)}$	$p \text{ (Pa)}$
0.05	343
0.15	1029
0.25	1372
0.35	2401
0.45	3087
0.55	4116
0.65	2744
0.75	1372
0.85	686

Use the tabulated pressure, position data to estimate the magnitude and position of the equivalent force.

Solution 2

w values can be calculated from pressure values, as before:

$$w = p \cdot y$$

$$\begin{aligned}
 & \mathbf{x} := \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \\ 0.35 \\ 0.45 \\ 0.55 \\ 0.65 \\ 0.75 \\ 0.85 \end{pmatrix} \text{ m} & \mathbf{p} := \begin{pmatrix} 343 \\ 1029 \\ 1372 \\ 2401 \\ 3087 \\ 4116 \\ 2744 \\ 1372 \\ 686 \end{pmatrix} \text{ Pa} \\
 & y := 3.66 \text{ m} \\
 & \mathbf{w} := \mathbf{p} \cdot \mathbf{y} \\
 & \mathbf{w} = \begin{pmatrix} 1.255 \cdot 10^3 \\ 3.766 \cdot 10^3 \\ 5.022 \cdot 10^3 \\ 8.788 \cdot 10^3 \\ 1.13 \cdot 10^4 \\ 1.506 \cdot 10^4 \\ 1.004 \cdot 10^4 \\ 5.022 \cdot 10^3 \\ 2.511 \cdot 10^3 \end{pmatrix} \text{ Pa} \cdot \text{m}
 \end{aligned}$$

The integral for force is approximated as a summation:

$$F_R \approx \sum_{i=1}^N w_i \Delta x$$

where Δx is the width of a stack (0.10 m) and N is the number of stacks (9).

$$\begin{aligned}
 & \Delta x := 0.10 \text{ m} \\
 & F_R := \sum_{i=1}^9 w_i \Delta x \\
 & F_R = 6277 \text{ N}
 \end{aligned}$$

Note: The matrix index ORIGIN was set to 1 (rather than Mathcad's default value of 0) for this example using Math/Options/Array Origin.

Similarly, the centroid is approximated as follows:

$$x_{loc} := \frac{\sum_{i=1}^9 x_i w_i}{\sum_{i=1}^9 w_i}$$

$$x_{loc} = 0.49\text{m}$$

Annotated Mathcad Worksheets

Solution 1

Solution Using Loading Function

First, the polynomial coefficients for the pressure function are entered...

$$b := \begin{pmatrix} 1.071 \cdot 10^3 \\ -1.681 \cdot 10^4 \\ 1.191 \cdot 10^5 \\ -1.991 \cdot 10^5 \\ 9.591 \cdot 10^4 \end{pmatrix}$$

$$p(x) := (b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4) \quad \text{Pascals}$$

The $w(x)$ function is declared using the $p(x)$ function and the width of the boards, y .

$$y := 3.66 \quad \text{meters}$$

$$w(x) := p(x) y \quad \text{N/m}$$

The magnitude of the resultant force is calculated.

$$F_R := \int_{0.05}^{0.85} w(x) dx$$

$$F_R = 6.237 \cdot 10^3 \quad \text{Newtons}$$

The location of the resultant force is calculated.

$$x_{loc} := \frac{\int_{0.05}^{0.85} x w(x) dx}{\int_{0.05}^{0.85} w(x) dx}$$

$$x_{loc} = 0.496 \quad \text{meters}$$

Solution 2

Solution Using Tabulated Data

$$x := \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \\ 0.35 \\ 0.45 \\ 0.55 \\ 0.65 \\ 0.75 \\ 0.85 \end{pmatrix} \text{ m} \quad p := \begin{pmatrix} 343 \\ 1029 \\ 1372 \\ 2401 \\ 3087 \\ 4116 \\ 2744 \\ 1372 \\ 686 \end{pmatrix} \text{ Pa}$$

$$y := 3.66 \text{ m}$$

Calculate w values from pressure values using the length of the boards.

$$w := p \cdot y$$

$$w = \begin{pmatrix} 1.255 \cdot 10^3 \\ 3.766 \cdot 10^3 \\ 5.022 \cdot 10^3 \\ 8.788 \cdot 10^3 \\ 1.13 \cdot 10^4 \\ 1.506 \cdot 10^4 \\ 1.004 \cdot 10^4 \\ 5.022 \cdot 10^3 \\ 2.511 \cdot 10^3 \end{pmatrix} \text{ Pa} \cdot \text{m}$$

Calculate the magnitude of the resultant force (the use of a summation instead of an integration makes this an approximate result).

$$\Delta x := 0.10 \text{ m}$$

$$F_R := \sum_{i=1}^9 w_i \cdot \Delta x$$

$$F_R = 6277 \text{ N}$$

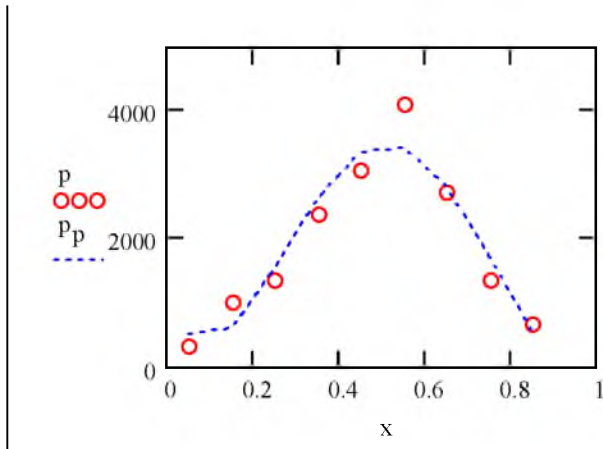
Calculate the location of the resultant force

$$x_{loc} := \frac{\sum_{i=1}^9 x_i \cdot w_i \cdot \Delta x}{\sum_{i=1}^9 w_i \cdot \Delta x}$$

$$x_{loc} = 0.49 \text{ m}$$

A Final Note

Both of these solutions are approximations. The second method approximates the integrals using summations, while the first method approximates the solution because the polynomial regression is a “best fit” but not a perfect fit to the data. In the graph shown here, the circles represent the tabulated values in Example 2, while the pressures predicted by the polynomial are indicated by the dashed line.



7

Equilibrium of a Rigid Body

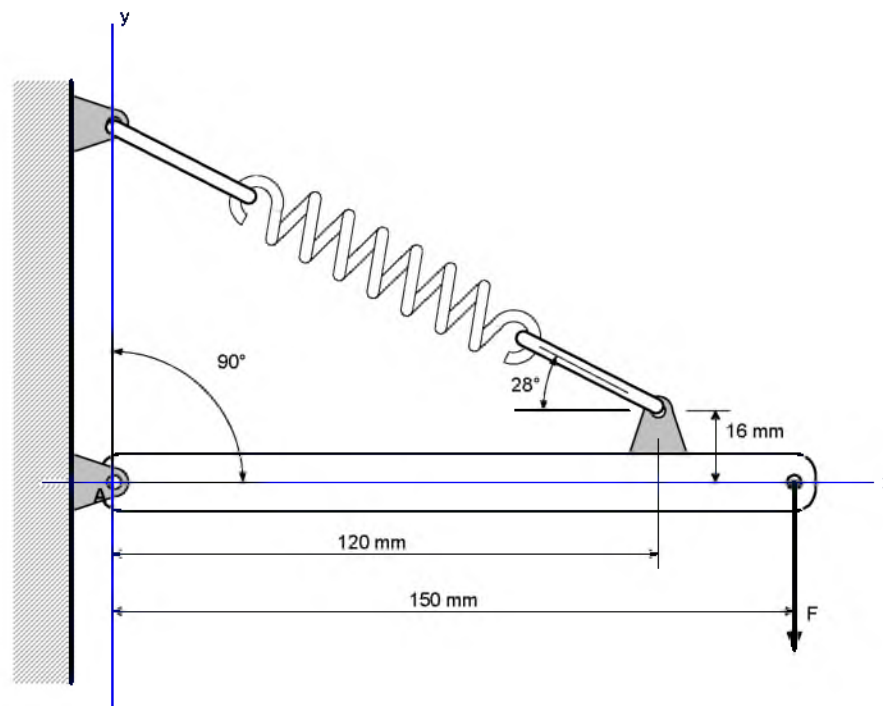
Ref: Hibbeler § 5.1-5.3, Bedford & Fowler: Statics § 5.1-5.2

When forces are applied to *rigid bodies*, the conditions of *equilibrium* can be used to determine any unknown forces, and the *reactions* at the supports.

Example 1: Spring Support

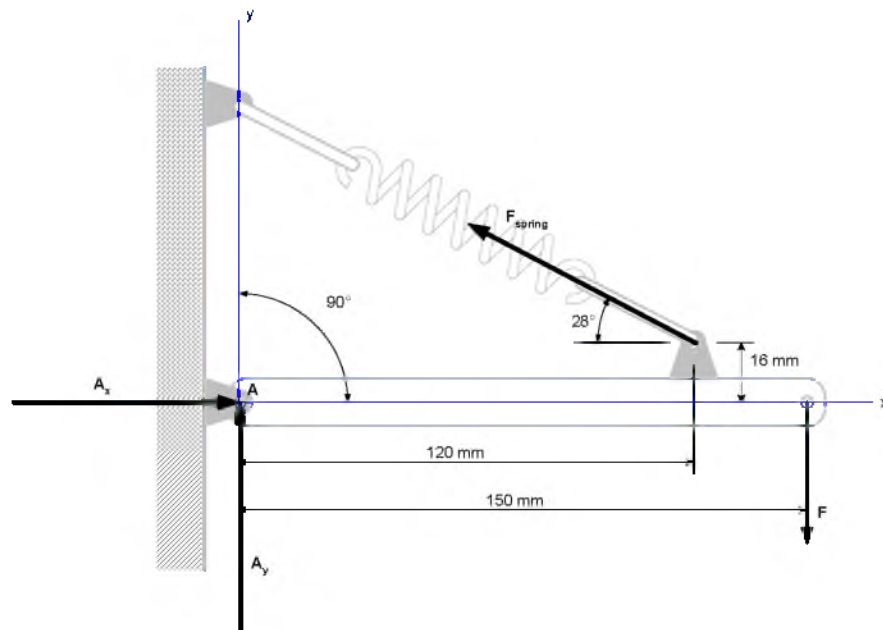
A force, F , is applied to the end of a bar to stretch a spring until the bar is horizontal. If the spring constant is $k = 600 \text{ N/m}$ and the unstretched length of the spring assembly (spring and connecting links) is 75 mm , and ignoring the mass of the bar, determine:

- the magnitude of the applied force, F , and
- the reactions at point A.



Solution

First, a free-body diagram is drawn.



We have assumed that the reactions at A will include both x and y components, and will calculate the actual values of A_x and A_y as part of the solution process.

First, a little trigonometry is required to obtain the extended length of the spring assembly.

$$L_{\text{ext}} := \frac{120 \text{ mm}}{\cos(28^\circ)}$$

$$L_{\text{ext}} = 136 \text{ mm}$$

Then, we can calculate the actual extension of the spring...

$$L_{\text{unstretched}} := 75 \text{ mm}$$

$$L_{\text{stretch}} := L_{\text{ext}} - L_{\text{unstretched}}$$

$$L_{\text{stretch}} = 61 \text{ mm}$$

...and the spring force, F_{spring} .

$$k_{\text{spring}} := 600 \frac{\text{N}}{\text{m}}$$

$$F_{\text{spring}} := k_{\text{spring}} \cdot L_{\text{stretch}}$$

$$F_{\text{spring}} = 36.5 \text{ N}$$

Next, we calculate the x and y components of the spring force, using the angle from the positive x axis (not just 28°) so that the direction of the force is accounted for.

$$\theta := (180 - 28) \text{ deg} \quad \ll \text{angle from } +x \text{ axis}$$

$$F_{sp_x} := F_{spring} \cos(\theta) \quad F_{sp_x} = 32.3 \text{ N} \quad \text{acting in } -x \text{ direction}$$

$$F_{sp_y} := F_{spring} \sin(\theta) \quad F_{sp_y} = 17.2 \text{ N}$$

At equilibrium the sum of the moments at A must be zero. We can use this to solve for the applied force, F.

$$|F_{sp_x}| (160 \text{ mm}) + F_{sp_y} (120 \text{ mm}) + F (150 \text{ mm}) = 0$$

Solve for F...

$$F := - \frac{|F_{sp_x}| (160 \text{ mm}) + F_{sp_y} (120 \text{ mm})}{150 \text{ mm}}$$

$$F = 17.2 \text{ N} \quad \text{acting to cause clockwise rotation}$$

Notes:

- The first equation in the last example block states the equilibrium relationship, but was not actually solved by Mathcad. Only the equation after "Solve for F..." was actually used to calculate F.
- The absolute value operator was used on F_{sp_x} since the direction of rotation was accounted for as the equilibrium equation was written. Here, counter-clockwise rotation was assumed positive.

The equilibrium relationships for the x and y components of force can be used to determine the reactions at A. First, x-component equilibrium requires that the sum of the x components of force be zero. This relationship is used to solve for A_x .

$$A_x + F_{sp_x} = 0$$

Solve for A_x .

$$A_x := -F_{sp_x}$$

$$A_x = 32.267 \text{ N} \quad \text{acting in } +x \text{ direction}$$

Finally, the equilibrium relationship for the sum of the y components of force is used to calculate A_y .

$$A_y + F_{sp_y} + F = 0$$

Solve for A_y .

$$A_y := -F_{sp_y} - F$$

$$A_y = 0.0105 \text{ N}$$

Annotated Mathcad Worksheet

Spring Support Problem

Find the total extended length of spring assembly.

$$L_{\text{ext}} := \frac{120 \text{ mm}}{\cos(28^\circ)}$$

$$L_{\text{ext}} = 136 \text{ mm}$$

Calculate the spring's extension.

$$L_{\text{unstretched}} := 75 \text{ mm}$$

$$L_{\text{stretch}} := L_{\text{ext}} - L_{\text{unstretched}}$$

$$L_{\text{stretch}} = 61 \text{ mm}$$

Calculate spring force.

$$k_{\text{spring}} := 600 \frac{\text{N}}{\text{m}}$$

$$F_{\text{spring}} := k_{\text{spring}} \cdot L_{\text{stretch}}$$

$$F_{\text{spring}} = 36.5 \text{ N}$$

Find x and y components of spring force.

$$\theta := (180 - 28)^\circ \quad \ll \text{angle from } +x \text{ axis}$$

$$F_{\text{sp}_x} := F_{\text{spring}} \cdot \cos(\theta) \quad F_{\text{sp}_x} = 32.3 \text{ N} \quad \text{acting in } -x \text{ direction}$$

$$F_{\text{sp}_y} := F_{\text{spring}} \cdot \sin(\theta) \quad F_{\text{sp}_y} = 17.2 \text{ N}$$

Use the sum of the moments about A to find applied force F.

At equilibrium, the sum of the moments at A must be zero. Here, counter-clockwise rotation is considered positive.

$$F_{\text{sp}_x} \cdot (16 \text{ mm}) + F_{\text{sp}_y} \cdot (12 \text{ mm}) + F \cdot (15 \text{ mm}) = 0$$

Solve for F...

$$F := - \frac{F_{\text{sp}_x} \cdot (16 \text{ mm}) + F_{\text{sp}_y} \cdot (12 \text{ mm})}{15 \text{ mm}}$$

$$F = 17.2 \text{ N} \quad \text{acting to cause clockwise rotation}$$

Use x and y component force balances to solve for the reactions at A.

At equilibrium, the sum of the x components of force must be zero.

$$A_x + F_{sp_x} = 0$$

Solve for A_x .

$$A_x := -F_{sp_x}$$

$$A_x = 32.267\text{N} \quad \text{acting in } +x \text{ direction}$$

At equilibrium, the sum of the y components of force must be zero.

$$A_y + F_{sp_y} + F = 0$$

Solve for A_y .

$$A_y := -F_{sp_y} - F$$

$$A_y = 0.0105\text{N}$$

8

Dry Friction

Ref: Hibbeler § 8.2, Bedford & Fowler: Statics § 9.1

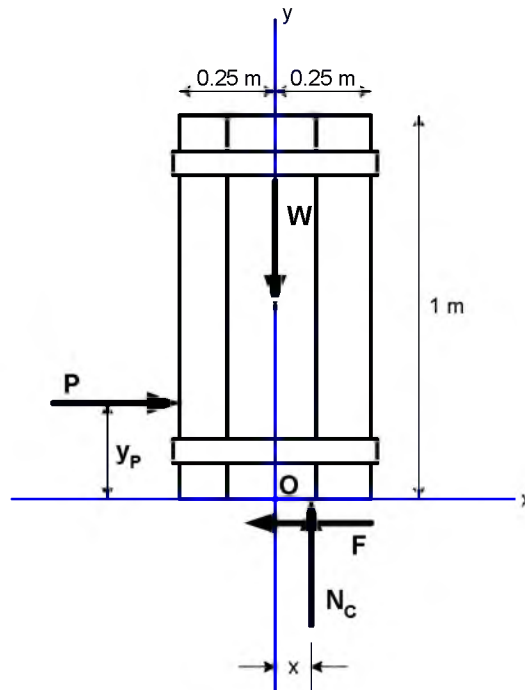
To move a heavy crate, such as the one illustrated in the example below, you have to push on it. If you push hard enough to overcome *friction*, the crate will slide along the floor. But there is the danger that the crate will tip, especially if you push too high on the side of the crate. To better understand the calculations involved with this dry friction problem, and to see how you can check for tipping, we will solve the “sliding a crate” problem for several cases, varying the push height between 0.2 and 1 m, and the magnitude of the pushing force between 100 and 120 N.

Example: Pushing a Crate

A force, P , is applied to the side of a crate at height y_P from the floor. The crate weighs 35 kg, and is 0.5 m wide and 1.0 m in height. The coefficient of static friction is 0.32. For each of the test cases tabulated below, determine:

- the magnitude of the friction force, F ,
- the magnitude of the resultant normal force, N_C , and
- the distance, x , (measured from the origin indicated in the illustration below) at which the resultant normal force acts.

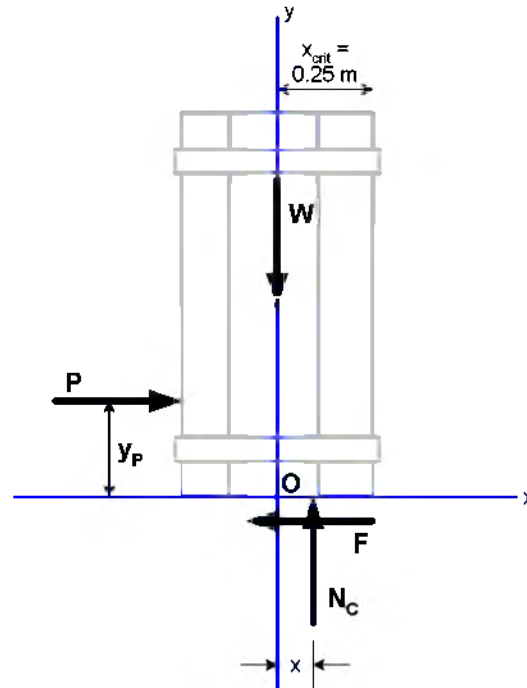
Once these results have been calculated, check for tipping and/or slipping.



Test Cases	P	y_P
1	100 N	0.2 m
2	100 N	0.4 m
3	100 N	0.8 m
4	100 N	1.0 m
5	120 N	0.2 m
6	120 N	0.4 m
7	120 N	0.8 m
8	120 N	1.0 m

Solution, Case 1

First, a free-body diagram is drawn.



To start a Mathcad worksheet, we assign the values given in the problem statement to variables.

Data from the problem statement...

for all cases:	$M := 350 \text{ kg}$	$\mu_s := 0.32$	$x_{\text{crit}} := 0.25 \text{ m}$
for specific case:	$\text{case} := "1"$	$P := 100 \text{ N}$	$y_P := 0.2 \text{ m}$

Then we calculate the weight of the crate.

$$W := M \cdot g$$
$$W = 343.2 \text{ N}$$

Note: The acceleration due to gravity, g , is a predefined variable in Mathcad.

Next, we use the equilibrium condition that the sum of the forces in the x direction must be zero. (This condition holds if the crate is not sliding or tipping.)

$$P + F = 0$$

so...

$$F := -P$$
$$F = 100 \text{ N} \quad \text{acts in the } -x \text{ direction}$$

Then, we use the equilibrium relationship for the y-components of force to determine the resultant normal force, N_C .

$$W + N_C = 0$$

so...

$$N_C := -W$$

$$N_C = 343.2 \text{ N}$$

The last equilibrium relationship, that the sum of the moments about O must be zero, can be used to determine the location at which N_C acts, called x in the illustration above.

$$-P y_P + N_C x = 0$$

so...

$$x := \frac{P y_P}{N_C}$$

$$x = 0.058 \text{ m}$$

At this point the equilibrium calculations are complete. We can solve for F_{\max} , the maximum force that can be applied before overcoming friction. If the frictional force, F , exceeds F_{\max} , then the crate would slip. (As it begins to slip, the equilibrium relationships are no longer valid.)

$$F_{\max} := \mu_s N_C$$

$$F_{\max} = 109.8 \text{ N}$$

$$\text{slipCheck} := \text{if}(|F| > F_{\max}, \text{"slipping"}, \text{"not slipping"})$$

$$\text{slipCheck} = \text{"not slipping"}$$

Note: The absolute value operator was used on F since both the magnitude and direction of F (the minus sign) were determined using the equilibrium relationship. Only the magnitude is used to test for slippage.

Finally, we can check for tipping by testing to see if the calculated x is beyond the edge of the crate (x_{crit}).

$$\text{tipCheck} := \text{if}(x > x_{\text{crit}}, \text{"tipping"}, \text{"not tipping"})$$

$$\text{tipCheck} = \text{"not tipping"}$$

Note: The if() functions are certainly not required here – the test could be performed by inspection – but it is convenient to let Mathcad do the checking for each of the eight cases.

From these test we can see that a push of 100 N applied at a height of 0.2 m will not cause the crate to tip, but it will not cause the crate to move (slip), either.

Solution, Case 2

The only change between Case 1 and Case 2 is the height at which the push is applied. In Case 2 $y_P = 0.4 \text{ m}$. This value is change at the top of the worksheet, and Mathcad automatically recalculates the remaining calculations. The complete, annotated worksheet for Case 2 is shown below.

Annotated Mathcad Worksheet – Case 2

Data from the problem statement...

for all cases: $M := 35 \text{ kg}$ $\alpha_s := 0.32$ $x_{\text{crit}} := 0.25 \text{ m}$

for specific case: $\text{case} := "2"$ $P := 100 \text{ N}$ $y_P := 0.4 \text{ m}$

Calculate the weight of the crate.

$$W := M \cdot g$$

$$W = 343.2 \text{ N}$$

EQUILIBRIUM: The sum of the x components of force is zero.

$$P + F = 0$$

so...

$$F := -P$$

$$F = 100 \text{ N} \quad \text{acts in the -x direction}$$

EQUILIBRIUM: The sum of the y components of force is zero.

$$W + N_C = 0$$

so...

$$N_C := -W$$

$$N_C = 343.2 \text{ N}$$

EQUILIBRIUM: The sum of the moments about O is zero.

$$-P y_P + N_C x = 0$$

so...

$$x := \frac{P y_P}{N_C}$$

$$x = 0.117 \text{ m}$$

Checking the results...

$$F_{\text{max}} := \alpha_s N_C$$

$$F_{\text{max}} = 109.8 \text{ N}$$

$$\text{slipCheck} := \text{if}(|F| > F_{\text{max}}, \text{"slipping"}, \text{"not slipping"})$$

$$\text{slipCheck} = \text{"not slipping"}$$

$$\text{tipCheck} := \text{if}(x > x_{\text{crit}}, \text{"tipping"}, \text{"not tipping"})$$

$$\text{tipCheck} = \text{"not tipping"}$$

Summary of Results for All Eight Cases

By repeatedly changing the assigned values for P and y_P in the worksheet, the results for all cases can be determined. Those results are summarized here. The value $N_C = 343.2 \text{ N}$ is not case dependent in this problem.

<i>Test Cases</i>	<i>P</i>	<i>y_P</i>	<i>F</i>	<i>x</i>	<i>Slip?</i>	<i>Tip?</i>
1	100 N	0.2 m	-100 N	0.058 m	No	No
2	100 N	0.4 m	-100 N	0.117 m	No	No
3	100 N	0.8 m	-100 N	0.233 m	No	No
4	100 N	1.0 m	-100 N	0.291 m*	No	Yes
5	120 N	0.2 m	-120 N	0.070 m	Yes	No
6	120 N	0.4 m	-120 N	0.140 m	Yes	No
7	120 N	0.8 m	-120 N	0.280 m*	Yes	Yes
8	120 N	1.0 m	-120 N	0.350 m*	Yes	Yes

* The calculated result for x has no meaning except to indicate that the crate is tipping if $x > x_{crit}$.

9

Finding the Centroid of Volume

Ref: Hibbeler § 9.2, Bedford & Fowler: Statics § 7.4

The *centroid of volume* is the geometric center of a body. If the density is uniform throughout the body, then the *center of mass* and *center of gravity* correspond to the centroid of volume. The definition of the centroid of volume is written in terms of ratios of integrals over the volume of the body.

$$\bar{x} = \frac{\int_V x \, dV}{\int_V dV} \quad \bar{y} = \frac{\int_V y \, dV}{\int_V dV} \quad \bar{z} = \frac{\int_V z \, dV}{\int_V dV}$$

Either analytical or numerical integration methods can be used to evaluate these integrals and compute the centroid of volume for the body.

The integrals over volume take slightly different forms depending on the coordinate system you use.

Cartesian Coordinates

$$\int_V dV = \int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} dx \, dy \, dz$$

or, in Mathcad's form (integral limits in the same order as the d's)

$$\int_V dV = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} dx \, dy \, dz$$

Cylindrical Coordinates

$$\int_V dV = \int_{z_1}^{z_2} \int_{\phi_1}^{\phi_2} \int_{r_1}^{r_2} r \, dr \, d\phi \, dz$$

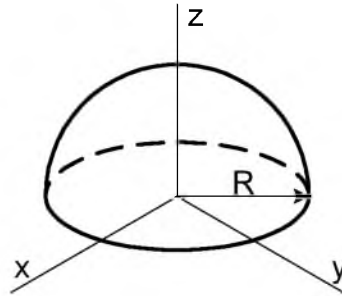
Spherical Coordinates

$$\int_V dV = \int_{\phi_1}^{\phi_2} \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} r^2 \sin \theta \, dr \, d\theta \, d\phi$$

These integrals can be evaluated using analytical or numerical integration techniques. Both will be illustrated here.

Example: Centroid of a Hemisphere

Find the centroid of volume for a hemisphere of radius $R = 7$ cm.



Note: This simple example will allow us to check our results against published values. For example, Hibbeler shows (inside back cover) that the centroid for a hemisphere resting on the plane formed by the x and y axes to be located at $x = 0$, $y = 0$, $z = 3/8 R$.

Solution: Analytical Integration

Mathematically, the integration for volume of a hemisphere (the denominator of the volume centroid equation) looks like this:

$$V = \int_V dV = \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{2\pi} \int_{r=0}^R r^2 \sin \phi \, dr \, d\phi \, d\theta$$

In Mathcad, this triple integral can be written as

$$\int_0^{\pi/2} \int_0^{2\pi} \int_0^R r^2 \sin(\phi) \, dr \, d\phi \, d\theta$$

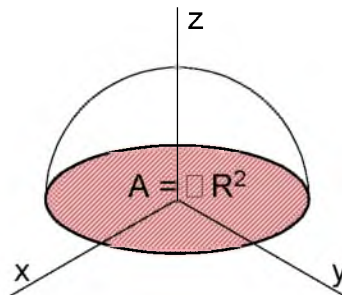
and evaluated using the symbolic math processor.

$$\int_0^{\pi/2} \int_0^{2\pi} \int_0^R r^2 \sin(\phi) \, dr \, d\phi \, d\theta = \frac{2}{3} \pi R^3$$

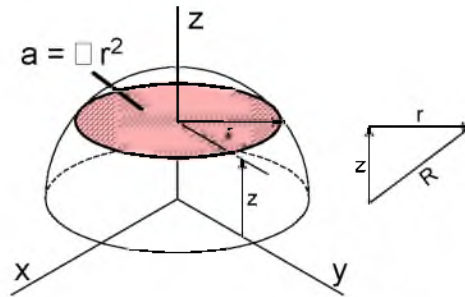
The volume of the hemisphere is $2/3 \pi R^3$ – hardly a surprise.

Alternatively, the volume can be calculated by using the equation for the area of a circle, and integrating in only one direction, z .

To use this approach, first note that the area of the base is easily calculated, as $A = \pi R^2$.



The area of the circle formed by any x-y plane through the hemisphere is calculated as $a = \pi r^2$.



where the value of r depends on z . The relationship between r and z is readily determined, since r and z are two sides of a right triangle in which the hypotenuse is the radius of the hemisphere, R .

$$r = \sqrt{R^2 - z^2}$$

We then integrate the area of the circle from $z = 0$ to $z = R$.

$$V = \int_V dV = \int_{z=0}^R a \, dz = \int_{z=0}^R (\pi r^2) \, dz = \int_{z=0}^R \pi (R^2 - z^2) \, dz$$

Mathcad will do this calculation as well, with the same result.

$$\left| \int_0^R \pi (R^2 - z^2) \, dz \right| = \frac{2}{3} \pi R^3$$

The denominator of the centroid equation is now known, let's work on the numerator...

The numerator of the volume centroid equation is just like the denominator, except for the extra coordinate direction.

$$\bar{z} = \frac{\int_V z \, dV}{\int_V dV}$$

To calculate the volume centroid of the hemisphere, simply include the extra z in the numerator's integral.

$$\left| \frac{\int_0^R z \pi (R^2 - z^2) \, dz}{\int_0^R \pi (R^2 - z^2) \, dz} \right| = \frac{3}{8} R$$

We obtained the expected result, $\bar{z} = \frac{3}{8} R$.

Note: Since the hemisphere exhibits symmetry in the x- and y-coordinate directions, we only need to calculate \bar{z} . By symmetry, \bar{x} and \bar{y} are zero.

Mathcad Worksheet

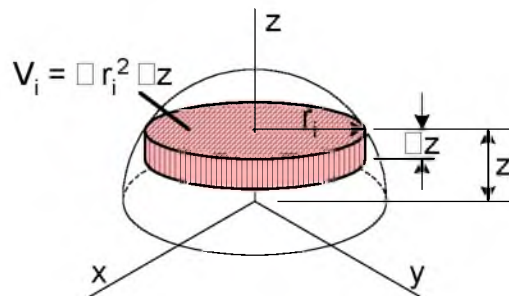
$$\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^R r^3 \sin(\theta) dr d\theta d\phi$$

$$\int_0^R \pi (R^2 - z^2) dz$$

$$\frac{\int_0^R z \pi (R^2 - z^2) dz}{\int_0^R \pi (R^2 - z^2) dz}$$

Solution: Numerical Integration

To approximate the result using numerical integration methods, rewrite the integral as a sum and the dz as a Δz . Graphically (for the volume calculation in the denominator) this is equivalent to approximating the volume of the hemisphere by a series of stacked disks of thickness Δz . One such disk is shown in the figure below.



Here, the value of r at the current (point "i") value of z has been used to calculate the volume of the disk. Since $R = 7$ cm in this example, we might try a Δz of 1 cm, and calculate the volumes of seven disks ($N = 7$). For each disk, the value of z_i can be calculated using

$$z_i = i \Delta z$$

The value of the corresponding radius is then

$$r_i = \sqrt{R^2 - z_i^2}$$

The formula for volume can then be written in terms of z , as

$$V_i = \pi (R^2 - z_i^2) \Delta z$$

The sum of all of the volumes of all seven disks is an approximation of the volume of the hemisphere.

$$V \approx \sum_{i=1}^N V_i = \sum_{i=1}^N \pi [R^2 - (i \Delta z)^2] \Delta z$$

In Mathcad, this calculation looks like this:

$$\begin{aligned} R &:= 7 \text{ cm} \\ N &:= 7 \\ \Delta z &:= \frac{R}{N} \\ \Delta z &= 1 \text{ cm} \\ V &:= \sum_{i=1}^N \pi [R^2 - (i \Delta z)^2] \Delta z \\ V &= 637.743 \text{ cm}^3 \end{aligned}$$

We can see how good (or bad) the approximation is by calculating the actual volume of the hemisphere.

$$\begin{aligned} V_{\text{actual}} &:= \frac{2}{3} \pi R^3 \\ V_{\text{actual}} &= 718.378 \text{ cm}^3 \end{aligned}$$

The approximation using only seven disks is not too good. If we use more disks, say $N = 70$, the approximation of the volume is quite a bit better.

$$\begin{aligned} R &:= 7 \text{ cm} \\ N &:= 70 \\ \Delta z &:= \frac{R}{N} \\ \Delta z &= 0.1 \text{ cm} \\ V &:= \sum_{i=1}^N \pi [R^2 - (i \Delta z)^2] \Delta z \\ V &= 710.644 \text{ cm}^3 \end{aligned}$$

To calculate the centroid using numerical methods, simply replace the ratio of integrals by the numerical approximation:

$$\bar{z} = \frac{\int_V z \, dV}{\int_V dV} = \frac{\sum_{i=1}^N z \left[R^2 - (i \Delta z)^2 \right] \Delta z}{\sum_{i=1}^N \left[R^2 - (i \Delta z)^2 \right] \Delta z}$$

Again, the extra “z” has been included in the numerator as (i Δz). In Mathcad, the calculation of the centroid looks like this:

$$z_{\text{bar}} := \frac{\sum_{i=1}^N (i \Delta z) \left[R^2 - (i \Delta z)^2 \right] \Delta z}{\sum_{i=1}^N \left[R^2 - (i \Delta z)^2 \right] \Delta z}$$

$$z_{\text{bar}} = 2.653 \text{ cm}$$

We can use the analytical result to check our answer.

$$z_{\text{bar_act}} := \frac{3}{8} R$$

$$z_{\text{bar_act}} = 2.625 \text{ cm}$$

Annotated Mathcad Worksheet

$R := 7.5 \text{ cm}$	<< define the system
$N := 70$	<< choose a number of disks
$\Delta z := \frac{R}{N}$	<< calculate the thickness of a disk
$\Delta z = 0.1 \text{ cm}$	
$z_{\text{bar}} := \frac{\sum_{i=1}^N (i \Delta z) \left[R^2 - (i \Delta z)^2 \right] \Delta z}{\sum_{i=1}^N \left[R^2 - (i \Delta z)^2 \right] \Delta z}$	<< calculate the centroid
$z_{\text{bar}} = 2.653 \text{ cm}$	
$z_{\text{bar_act}} := \frac{3}{8} R$	<< the analytical result can be used to check the numerical result if it is available
$z_{\text{bar_act}} = 2.625 \text{ cm}$	<< looks pretty good.

10

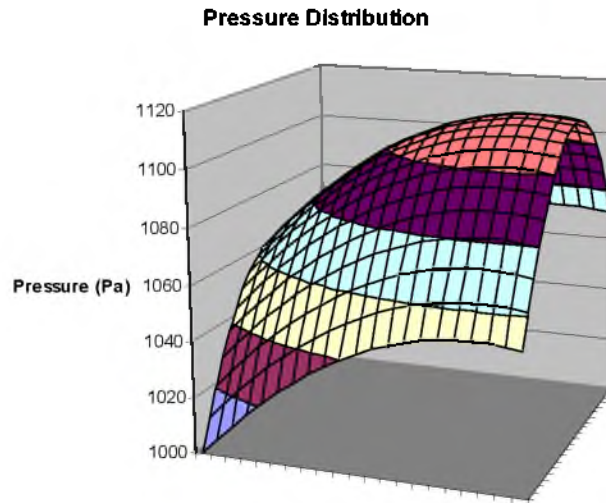
Resultant of a Generalized Distributed Loading

Ref: Hibbeler § 9.5, Bedford & Fowler: Statics § 7.4

When a load is continuously distributed across an area it is possible to determine the equivalent resultant force, and the position at which the resultant force acts. This is termed the *resultant of a generalized distributed loading*. This is the generalized case of the simple distributed loading considered earlier (#6).

Example: Load Distribution Expressed as a Function of x and y

The pressure on a surface is distributed as illustrated in the following surface plot:



This plot is described by the following function:

$$p(x, y) = 1000 + 230x + 210x^2 + 120y + 70y^2$$

The x and y values range from 0 to 1 meter. The pressure is expressed in Pa.

Determine the magnitude and location of the resultant force.

Solution

The magnitude of the resultant force is obtained by integration.

$$F_R = \int_A p(x, y) dA = \int_0^1 \int_0^1 p(x, y) dx dy$$

Mathcad can perform this integration.

$$p(x,y) := 1000 + 2300x + 2100x^2 + 1200y + 700y^2 \quad \text{Pascals}$$

$$F_R := \int_0^1 \int_0^1 p(x,y) \, dx \, dy$$

$$F_R = 1082 \quad \text{Newtons}$$

The location at which the resultant force acts is found by calculating the centroid of the volume defined by the distributed loading diagram.

$$x_{loc} := \frac{\int_0^1 \int_0^1 x p(x,y) \, dx \, dy}{\int_0^1 \int_0^1 p(x,y) \, dx \, dy} \quad \text{meters}$$

$$x_{loc} = 0.502 \quad \text{meters}$$

$$y_{loc} := \frac{\int_0^1 \int_0^1 y p(x,y) \, dx \, dy}{\int_0^1 \int_0^1 p(x,y) \, dx \, dy} \quad \text{meters}$$

$$y_{loc} = 0.504 \quad \text{meters}$$

Annotated Mathcad Worksheet

Resultant of a Generalized Distributed Loading

First, enter the pressure distribution function...

$$p(x,y) := 1000 + 2300x + 2100x^2 + 1200y + 700y^2 \quad \text{Pascals}$$

Calculate the magnitude of the resultant force.

$$F_R := \int_0^1 \int_0^1 p(x,y) \, dx \, dy$$

$$F_R = 1082 \quad \text{Newtons}$$

Calculate the x and y positions of the resultant force.

$$x_{loc} := \frac{\int_0^1 \int_0^1 x p(x,y) \, dx \, dy}{\int_0^1 \int_0^1 p(x,y) \, dx \, dy} \quad \text{meters}$$

$$x_{loc} = 0.502 \quad \text{meters}$$

$$y_{loc} := \frac{\int_0^1 \int_0^1 y p(x,y) \, dx \, dy}{\int_0^1 \int_0^1 p(x,y) \, dx \, dy} \quad \text{meters}$$

$$y_{loc} = 0.504 \quad \text{meters}$$

11

Calculating Moments of Inertia

Ref: Hibbeler § 10.1-10.2, Bedford & Fowler: Statics § 8.1-8.2

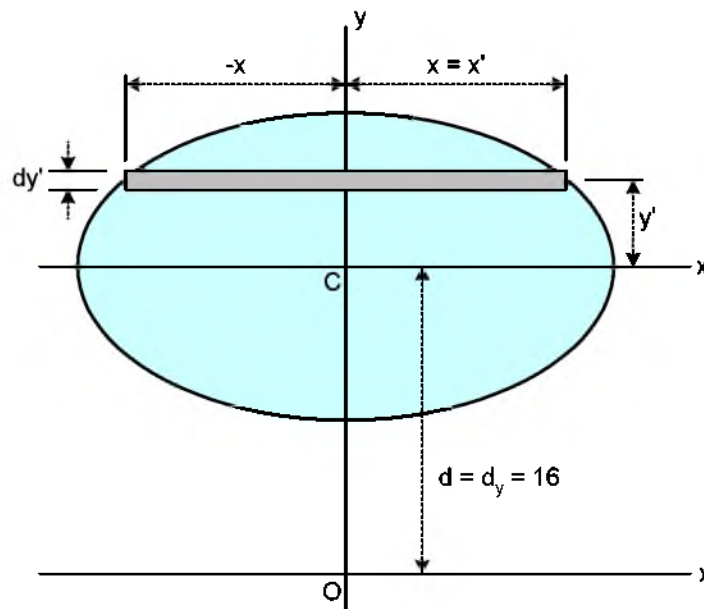
Calculating moments of inertia requires evaluating integrals. This can be accomplished either symbolically, or using numerical approximations. Mathcad's ability to integrate functions to generate numerical results is illustrated here.

Example: Moment of Inertia of an Elliptical Surface

Determine the moment of inertia of the ellipse illustrated below with respect to a) the centroidal x' axis, and b) the x axis.

The equation of the ellipse relative to centroidal axes is

$$\frac{x'^2}{8^2} + \frac{y'^2}{14^2} = 1$$



In this problem, x and y have units of cm.

Solution

The moment of inertia about the centroidal x axis is defined by the equation

$$I_{x'} = \int_A y'^2 dA$$

where dA is the area of the differential element indicated in the figure above.

$$dA = 2x dy'$$

So, the integral for moment of inertia becomes

$$I_{x'} = \int_A y'^2 \, dA$$

Furthermore, x (or x') can be related to y' using the equation of the ellipse.

Note: Because of the location of the axes, $x = x'$ in this example.

$$x = x' = \sqrt{8^2 \left(1 - \frac{y'^2}{14^2} \right)}$$

The equation for the moment of inertia becomes:

$$I_{x'} = \int_{-8}^8 y'^2 \, 2 \sqrt{8^2 \left(1 - \frac{y'^2}{14^2} \right)} \, dy'$$

Mathcad can perform this integration.

$$\begin{aligned} I_{x_prime} &:= \int_{-8}^8 y_{prime}^2 \, 2 \sqrt{8^2 \left(1 - \frac{y_{prime}^2}{14^2} \right)} \, dy_{prime} \\ I_{x_prime} &= 4890 \quad cm^4 \end{aligned}$$

The moment of inertia relative to the original x axis can be found using the *parallel-axis theorem*.

$$I_x = I_{x'} + A d_y^2$$

Where A is the area of the ellipse, and d_y is the displacement of the centroidal y axis from the original y axis.

The required area can be calculated by integration.

$$\begin{aligned} A &:= \int_{-8}^8 2 \sqrt{8^2 \left(1 - \frac{y_{prime}^2}{14^2} \right)} \, dy_{prime} \\ A &= 241.29 \quad cm^2 \end{aligned}$$

Then the moment of inertia about x can be determined.

$$\begin{aligned} d_y &:= 16 \quad cm \\ I_x &:= I_{x_prime} + A d_y^2 \\ I_x &= 66661 \quad cm^4 \end{aligned}$$

Annotated Mathcad Worksheet

Calculate the moment of inertia relative to the x' axis.

$$I_{x_prime} := \int_{-8}^8 y_{prime}^2 \cdot 2 \sqrt{8^2 \left(1 - \frac{y_{prime}^2}{14^2} \right)} dy_{prime}$$

$$I_{x_prime} = 4890 \quad cm^4$$

Calculate the area of the ellipse.

$$A := \int_{-8}^8 2 \sqrt{8^2 \left(1 - \frac{y_{prime}^2}{14^2} \right)} dy_{prime}$$

$$A = 241.29 \quad cm^2$$

Use the parallel-axis theorem to calculate the moment of inertia relative to the x axis.

$$d_y := 16 \quad cm$$

$$I_x := I_{x_prime} + A d_y^2$$

$$I_x = 66661 \quad cm^4$$

12

Curve-Fitting to Relate s-t, v-t, and a-t Graphs

Ref: Hibbeler § 12.3, Bedford & Fowler: Dynamics § 2.2

The relationships between a particle's position, s , velocity, v , and acceleration, a , over time are all related. When the relationship between a particle's velocity (for example) and time is described by a mathematical equation, that equation can be integrated to obtain the particle's position, and differentiated to determine the particle's acceleration. But what if the velocity vs. time relationship is described by a set of data values or a graph, rather than a mathematical equation? You can still obtain position and acceleration information, either directly from the data values (numerical approximations of integration and differentiation), or by fitting an equation to the data, then using calculus on the resulting equation. The curve-fitting approach will be demonstrated here.

Example: Fitting a Polynomial to Velocity Data

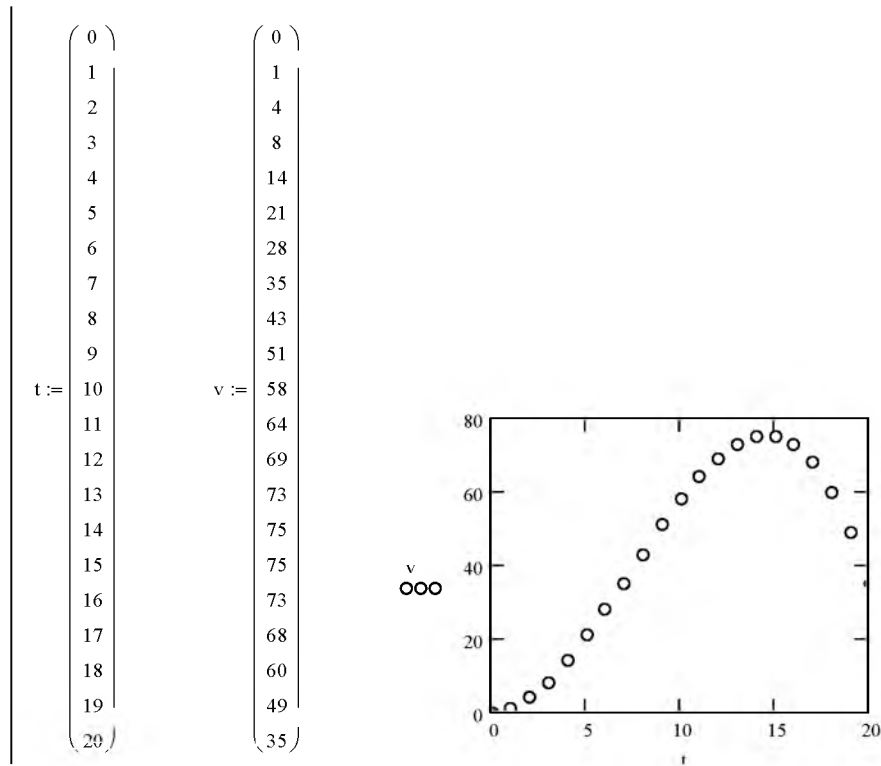
A driver on a straight track accelerates his car from 0 to 75 mph in 14 seconds, and then allows the car to begin to slow. A data recorder connected to the speedometer recorded the data shown below. Since the car is moving along a straight path, the car's speed and velocity are equal.

t (sec.)	v (mph)
0	0
1	1
2	4
3	8
4	14
5	21
6	28
7	35
8	43
9	51
10	58
11	64
12	69
13	73
14	75
15	75
16	73
17	68
18	60
19	49
20	35

Fit a curve to the v-t data, and then use the equation of the curve to plot s-t, v-t, and a-t graphs.

Solution – First Attempt

First, let's plot the data to see what we're working with.



The graph shows the increasing velocity for the first 14 seconds then the velocity drops off again. It looks like a smooth curve, and clean (not noisy) data.

A commonly used approach to fitting a curve to a set of data values is polynomial regression. We might try, for example, the following second-order polynomial.¹

$$v_p = b_0 + b_1 t + b_2 t^2$$

Mathcad provides a function for fitting linear models² such as this polynomial, the `linfit()` function. This function calculates the coefficient values that provide the best fit of the model equation to the data. The regression model is described by a vector showing the functionality of the independent variable (t , in this case) in each term of the model. For our second-order polynomial, the vector could be written as follows:

$$F(t) := \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}$$

Effectively, the $F(t)$ vector tells `linfit()` what is multiplying each of the b values it is trying to calculate.

Note: We want the velocity to be zero when $t = 0$. A way to force this to happen is to set $b_0 = 0$ in the $F(t)$ function.

¹ A second-order polynomial is not a good choice for this data. A parabola is a second-order equation, and the data shows a lot more curvature than a simple parabola. It will require a higher-order polynomial to adequately fit this data – but the goal of the first attempt is to demonstrate that the “best fit” polynomial may still be a very poor fit to the data.

² For curve-fitting, the variables are the model coefficients, the b values. All of the times are known from the data set. This polynomial is linear in the b values, so it is a linear model for curve fitting.

Then, `linfit()` uses the `t` and `v` values (as vectors), and the `F(t)` function shown above to calculate the `b` values that produce the “best” fit of the polynomial to the data.

$$b := \text{linfit}(t, v, F)$$

$$b = \begin{pmatrix} 0 \\ 8.227 \\ -0.271 \end{pmatrix}$$

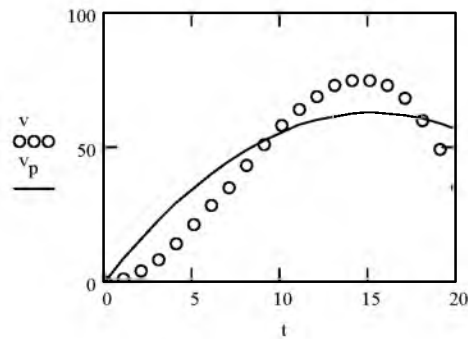
These coefficients tell us that the regression model is

$$v_p = 0 + 8.227 t - 0.271 t^2$$

The subscript `p` indicates that the equation is used to calculate predicted values of velocity.

The calculated coefficients provide the *best* possible fit of the model equation to the data, but that does not guarantee that the equation is a *good* fit. You should always calculate predicted values with the regression equation and compare them to the original data to visually determine if the regression equation is actually fitting the data.

$$v_p := b_0 + b_1 t + b_2 t^2 \quad v_p = \begin{pmatrix} 0 \\ 7.957 \\ 15.372 \\ 22.247 \\ 28.58 \\ 34.373 \\ 39.624 \\ 44.334 \\ 48.503 \\ 52.131 \\ 55.218 \\ 57.764 \\ 59.768 \\ 61.232 \\ 62.154 \\ 62.535 \\ 62.375 \\ 61.674 \\ 60.432 \\ 58.649 \\ 56.325 \end{pmatrix}$$



Clearly, the regression curve (solid line) is not doing a good job of fitting the data.

Solution – Second Attempt

We need a better regression equation. We'll try a third-order polynomial.

$$v_p = b_0 + b_1 t + b_2 t^2 + b_3 t^3$$

The `F(t)` vector gets another element...

$$F(t) := \begin{pmatrix} 0 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

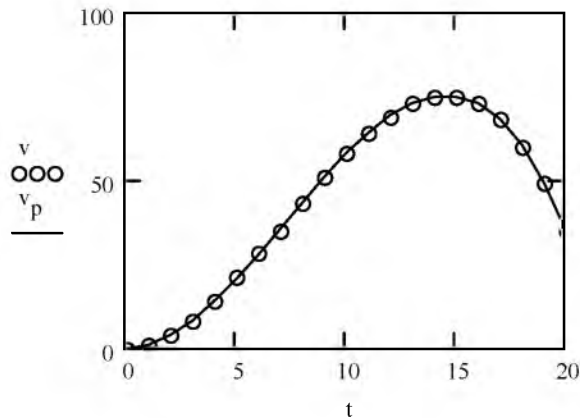
...and linfit() now finds four b values...

$$b := \text{linfit}(t, v, F)$$

$$b = \begin{pmatrix} 0 \\ 0.04 \\ 1.076 \\ -0.049 \end{pmatrix}$$

Again, we calculate predicted velocities to check the fit.

$$v_p := b_0 + b_1 t + b_2 t^2 + b_3 t^3$$



The new model is doing a good job of fitting the data. The regression equation that fits the data is

$$v_p = 0 + 0.04 t + 1.076 t^2 - 0.049 t^3$$

This equation can be used to generate the s-t and a-t graphs.

Mathcad can integrate the v_p equation to get position, s, and differentiate the equation for acceleration, a – but there's a problem...

...integration and differentiation requires Mathcad's symbolic math processor. We want to integrate and differentiate on t, but t has already been declared to be a vector of time values. We need an unassigned variable name for the symbolic processor, and t has already been used. To get around this, we will switch (briefly) to a new worksheet, where t is undefined.

New Worksheet

$$\frac{d}{dt}(b_0 + b_1 t + b_2 t^2 + b_3 t^3) = b_1 + 2b_2 t + 3b_3 t^2$$

$$\int (b_0 + b_1 t + b_2 t^2 + b_3 t^3) dt = b_0 t + \frac{1}{2} b_1 t^2 + \frac{1}{3} b_2 t^3 + \frac{1}{4} b_3 t^4$$

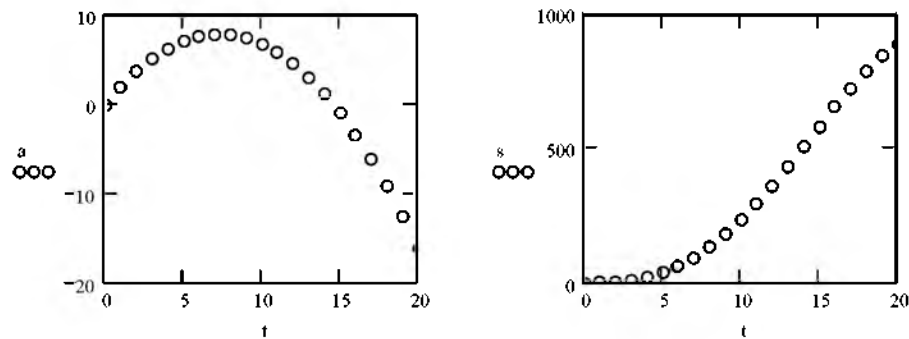
Note: When Mathcad integrates, it does not add the constant of integration. It effectively assumed the car was at $s = 0$ at $t = 0$.

With a bit of copying and pasting, the results of the calculus operations are returned to the original worksheet.

$$a := b_1 + 2b_2 t + 3b_3 t^2$$

$$s := b_0 t + \frac{1}{2} b_1 t^2 + \frac{1}{3} b_2 t^3 + \frac{1}{4} b_3 t^4$$

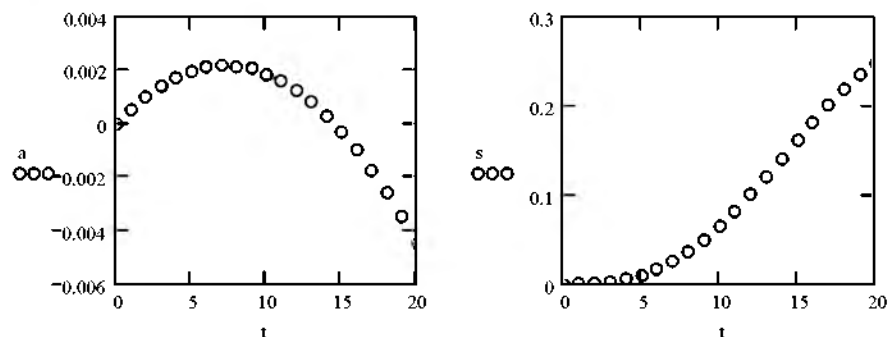
With the equations above, acceleration and position vectors a and s have been calculated by Mathcad. All that remains is to plot the results.



There are some strange units on the values plotted here. The acceleration has units of mi/sec hr, and the position, s , has units of mi sec/hr. We can clean this up a bit by building the unit conversions into the calculations for a and s .

$$a := \left(b_1 + 2b_2 t + 3b_3 t^2 \right) \frac{1}{3600}$$

$$s := \left(b_0 t + \frac{1}{2} b_1 t^2 + \frac{1}{3} b_2 t^3 + \frac{1}{4} b_3 t^4 \right) \frac{1}{3600}$$

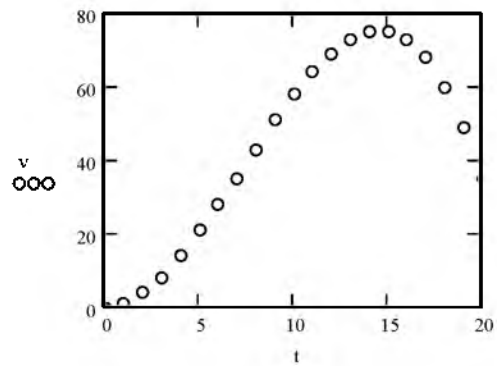


The units are now a [=] mi/sec² and s [=] miles.

Annotated Mathcad Worksheets

Define the time and velocity vectors, and plot the v-t graph.

$$t := \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 19 \\ 20 \end{pmatrix} \quad v := \begin{pmatrix} 0 \\ 1 \\ 4 \\ 8 \\ 14 \\ 21 \\ 28 \\ 35 \\ 43 \\ 51 \\ 58 \\ 64 \\ 69 \\ 73 \\ 75 \\ 75 \\ 73 \\ 68 \\ 60 \\ 49 \\ 35 \end{pmatrix}$$



First Solution Attempt - Second-Order Polynomial

Define the polynomial using the F(t) vector.

$$F(t) := \begin{pmatrix} 0 \\ t \\ t^2 \end{pmatrix}$$

Find the best-fit coefficients using the linfit() function.

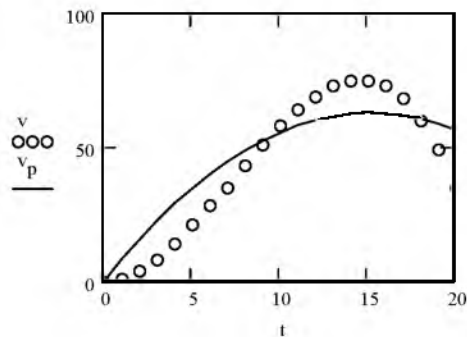
$$b := \text{linfit}(t, v, F)$$

Note: the linfit() function will not work with units.

$$b = \begin{pmatrix} 0 \\ 8.227 \\ -0.271 \end{pmatrix}$$

Calculate predicted velocity values (at each t value) to see if the regression equation actually fits the data.

$$v_p := b_0 + b_1 t + b_2 t^2$$



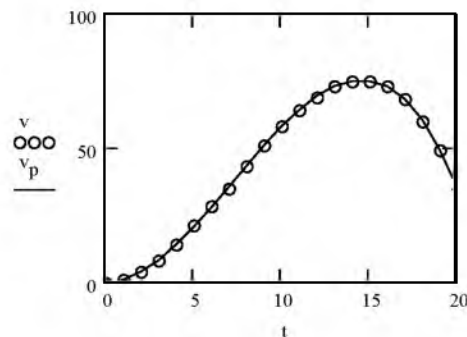
The predicted values do not agree with the data values - we need a better regression equation.

Try a third-order polynomial.

$$F(t) := \begin{pmatrix} 0 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

$$b := \text{linsfit}(t, v, F) \quad b = \begin{pmatrix} 0 \\ 0.04 \\ 1.076 \\ 0.049 \end{pmatrix}$$

$$v_p := b_0 + b_1 t + b_2 t^2 + b_3 t^3$$



This regression equation fits well.

Next, we switch to a different worksheet (where t is undefined) so that Mathcad can differentiate the regression function for acceleration and integrate it for position.

New Worksheet

$$\frac{d}{dt} (b_0 + b_1 t + b_2 t^2 + b_3 t^3) = b_1 + 2b_2 t + 3b_3 t^2$$

$$\int (b_0 + b_1 t + b_2 t^2 + b_3 t^3) dt = b_0 t + \frac{1}{2} b_1 t^2 + \frac{1}{3} b_2 t^3 + \frac{1}{4} b_3 t^4$$

Then, the results of each calculation are copied from the new worksheet and pasted back into the original worksheet as acceleration and position functions.

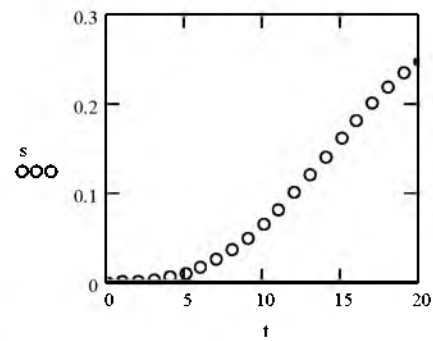
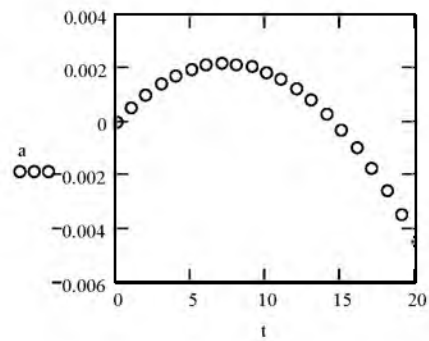
$$a := b_1 + 2b_2t + 3b_3t^2$$

$$s := b_0t + \frac{1}{2}b_1t^2 + \frac{1}{3}b_2t^3 + \frac{1}{4}b_3t^4$$

The units are simplified by building in the conversion from hours to seconds, and the results are plotted.

$$a := \left(b_1 + 2b_2t + 3b_3t^2 \right) \frac{1}{3600}$$

$$s := \left(b_0t + \frac{1}{2}b_1t^2 + \frac{1}{3}b_2t^3 + \frac{1}{4}b_3t^4 \right) \frac{1}{3600}$$



Curvilinear Motion: Rectangular Components

Ref: Hibbeler § 12.5, Bedford & Fowler: Dynamics § 2.3

A fixed x, y, z frame of reference is commonly used to describe the path of a particle in two situations:

1. The path "fits" the Cartesian coordinate system well (tends to follow straight lines, or simple paths.)
2. The path does not "fit" any commonly used coordinate system well.

The example used here falls into the latter category.

Example: Finding the Velocity and Acceleration of a Particle

The complex path of a particle for times between 0 and 10 seconds has been fit using multivariable regression to the following functions:

$$x(t) = 2 \cos(t)$$

$$y(t) = 1.3 + 2.7 t^2 - 0.031 t^3$$

$$z(t) = 2.4 + 0.14 t^3$$

The position of the particle at any point in time for which the functions are valid ($0 \leq t \leq 10$ sec.) can be determined as

$$\mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$$

Given these functions for x , y , and z determine the position, and the magnitudes of the velocity and acceleration of the particle at $t = 7$ sec.

Solution

The first thing we might want to do is graph this particle path, just to see what it looks like. Mathcad can help here. First, declare the three functions of time that describe the particle's path:

$$x(t) := 2 \cos(t)$$

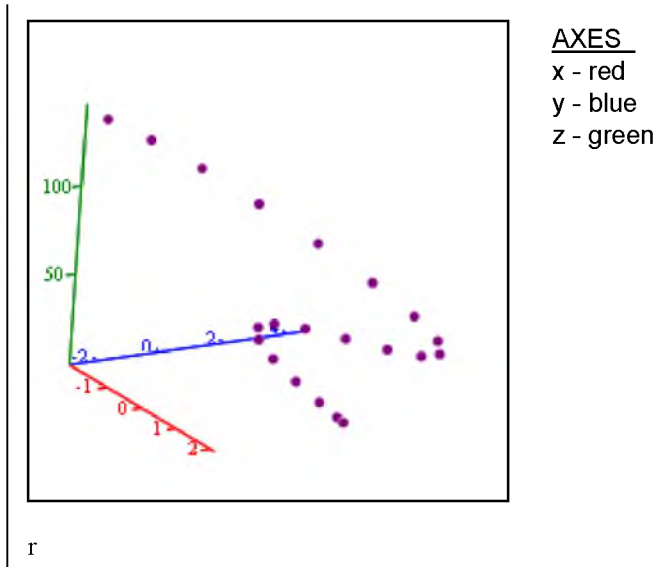
$$y(t) := 1.3 + 2.7 t^2 - 0.031 t^3$$

$$z(t) := 2.4 + 0.14 t^3$$

Then declare $\mathbf{r}(t)$ as a three-component matrix composed of these functions.

$$\mathbf{r}(t) := \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

Then ask Mathcad to plot the function by placing the function name, \mathbf{r} , in the placeholder of a 3-d scatter plot.



When Mathcad sees a function name in the placeholder, it tries to evaluate the function and plot the calculated result. Mathcad calls this a *QuickPlot*. Double-click on the graph to bring up the dialog box that allows you to set the graph's parameters. In the graph shown here, the QuickPlot range (of time values) was changed to 0 to 10 seconds, and the colors of the axes were changed. Then (after closing the dialog) the plot was rotated to make the path clearer by clicking and dragging the mouse on the graph.

Solving for Position

The position is described by function $r(t)$, so finding the position at $t = 7$ sec. Simply requires evaluating $r(t)$ at this time.

$$r(7) = \begin{pmatrix} 1.508 \\ 3.897 \\ 50.42 \end{pmatrix}$$

So, at $t = 7$ seconds, the particle is located at $x = 1.508$, $y = 3.897$, and $z = 50.42$.

Solving for the Magnitude of the Velocity

For velocity we need to take the derivative of the x , y , and z functions. Here, the derivatives obtained using the symbolic math process (to the right of the arrows) have been assigned to three new functions: $v_x(t)$, $v_y(t)$, and $v_z(t)$.

$$\begin{array}{ll} \frac{d}{dt}x(t) \rightarrow 2\sin(t) & v_x(t) := 2\sin(t) \\ \frac{d}{dt}y(t) \rightarrow .54t + 9.3 \cdot 10^{-2}t^2 & v_y(t) := .54t + 9.3 \cdot 10^{-2}t^2 \\ \frac{d}{dt}z(t) \rightarrow .42t^2 & v_z(t) := .42t^2 \end{array}$$

The magnitude of the velocity at $t = 7$ seconds can now be determined.

$$v = 20.637$$

To find the acceleration, we need to differentiate the v_x , v_y , and v_z functions with respect to time. The derivatives obtained using the symbolic math process (to the right of the arrows) have been assigned to three new functions: $a_x(t)$, $a_y(t)$, and $a_z(t)$.

$$a_x(t) := -2\cos(t)$$

$$a_v(t) := .54 \square .186 \mathbb{I}$$

$$a_Z(t) := .84\mathbb{I}$$

The magnitude of the acceleration at $t = 7$ seconds is

$a = 6.118$

Curvilinear Motion: Rectangular Components

$$x(t) := 2\cos(t)$$

$$z(t) := 2.4 + 0.14t^3$$

$$\mathbf{x}(t) := \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

AXES
x - red
y - blue
z - green

$$t(7) = \begin{pmatrix} 1.508 \\ 3.897 \\ 50.42 \end{pmatrix}$$
$$\begin{aligned} \frac{d}{dt}x(t) &= 2\sin(t) & v_x(t) &= 2\sin(t) \\ \frac{d}{dt}y(t) &= .54t + 9.3t^2 & v_y(t) &= .54 + 18.6t \\ \frac{d}{dt}z(t) &= .42t^2 & v_z(t) &= .84t \end{aligned}$$
$$v := \sqrt{v_x(7)^2 + v_y(7)^2 + v_z(7)^2}$$

$$v = 20.637$$

$\frac{d}{dt}v_x(t) = -2\cos(t)$	$a_x(t) := -2\cos(t)$
$\frac{d}{dt}v_y(t) = .54 + .1860000000000000000000$	$a_y(t) := .54 + .186$
$\frac{d}{dt}v_z(t) = .84$	$a_z(t) := .84$

$$a := \sqrt{a_X(7)^2 + a_Y(7)^2 + a_Z(7)^2}$$

$$a = 6.118$$

Curvilinear Motion, Motion of a Projectile

Ref: Hibbeler § 12.6, Bedford & Fowler: Dynamics § 2.3

Rectilinear motion refers to motion in a straight line. When a particle follows a non-straight path, it's motion is termed *curvilinear*. Projectile motion is typically curvilinear, although a projectile fired straight up (in the absence of a crosswind), or moving along a straight track would be rectilinear motion.

A projectile's motion can be broken down into three phases: an acceleration phase where the actuator (gun, catapult, golf club, etc.) gets the projectile moving. The second phase of motion is after the projectile leaves the actuator, when the only acceleration acting on it is the acceleration due to gravity.

Note: A common assumption that simplifies the problem considerably, but is not altogether accurate, is that the frictional drag between the projectile and the fluid through which it moves is negligible. This assumption is more reasonable for small, smooth, slow-moving particles through low-viscosity fluid than for large, irregularly shaped particles moving at high speeds through highly viscous fluids.

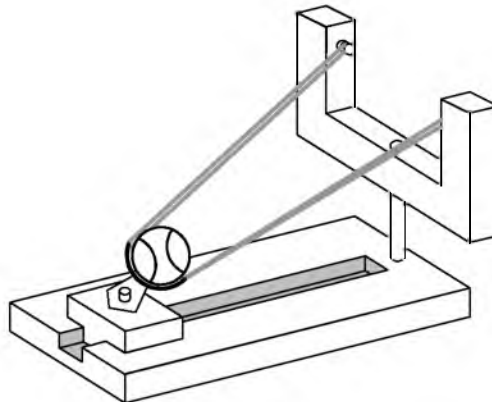
The third phase of a projectile's motion is after impact. The particle may roll, continue moving through a very different medium (water or earth), or break up. Here we will consider only the second phase of the projectile's motion: the projectile has already been accelerated by an actuator, and is flying through the air.

Example: Slingshot Contest

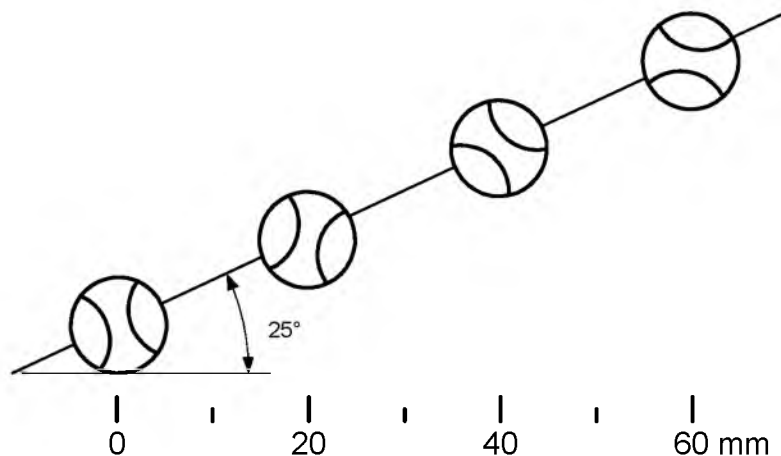
Projectile throwing contests are pretty common at engineering schools. There are several ways to run the contest: highest, farthest, most accurate, etc; and several ways to propel the projectile: slingshot, catapult, etc. This example focuses on a tennis ball slingshot competition.

<i>Tennis Ball Data (approximate)</i>	
Diameter	2.5 inches (6.4 cm)
Weight	2 oz (57 g)

A team of mechanical engineering students has built a slingshot that allows precise control of the pre-release tension and angle. They also connected the release mechanism to a digital camera to take a series of photos in 1 millisecond intervals just as the tennis ball is released. The photos are used to calculate the velocity (speed and angle) at which the tennis ball leaves the sling.



A test run with the camera operational gave the following set of photos (superimposed).



Part 1.

Determine:

- The initial velocity of the tennis ball as it leaves the sling.
- The predicted time of flight for the ball.
- The predicted horizontal travel distance for the ball.

Part 2.

The team that gets most balls into a basket set 30 meters from the launch site wins. If they can keep the initial speed constant (at the test run value of 22.1 m/s), what angle should they use to shoot the tennis balls into the basket?

Part 1. Solution

The initial velocity is calculated from the 60 mm horizontal travel distance observed in the photos using the camera's 1 millisecond interval between snapshots.

$x_{\text{test}} := 60 \text{ mm}$	<< horizontal travel distance for all four frames
$\theta := 25 \text{ deg}$	
$d_{\text{test}} := \frac{x_{\text{test}}}{\cos(\theta)}$	
$d_{\text{test}} = 66.2 \text{ mm}$	<< distance ball travelled in direction of motion (four frames)
$\Delta t_{\text{pic}} := 0.001 \text{ sec}$	
$\Delta t_{\text{test}} := 3 \Delta t_{\text{pic}}$	<< 3 time intervals between the four photos
$v_{\text{init}} := \frac{d_{\text{test}}}{\Delta t_{\text{test}}}$	
$v_{\text{init}} = 22.1 \frac{\text{m}}{\text{s}}$	

The horizontal and vertical components of the initial velocity will be useful for later calculations.

$$\begin{aligned}
 v_{y_init} &:= v_{init} \sin(\theta) \\
 v_{y_init} &= 9.3 \frac{\text{m}}{\text{s}} \quad \ll \text{y component of initial velocity} \\
 v_{x_init} &:= v_{init} \cos(\theta) \\
 v_{x_init} &= 20 \frac{\text{m}}{\text{s}} \\
 v_x &:= v_{x_init} \quad \ll \text{x component of velocity is constant (if air resistance is ignored)}
 \end{aligned}$$

The predicted time of flight can be calculated using

$$y = y_0 + v_{y_init} t_{\text{flight}} + \frac{1}{2} a_c t_{\text{flight}}^2$$

where a_c is the constant acceleration. In this problem, the acceleration is due to gravity and acts in the $-y$ direction, so $a_c = -g$.

If we assume that the flight is over a horizontal surface, then y at the end of the flight is zero. Another common assumption is to assume that y_0 is also zero. This is a reasonable assumption if the vertical position of the tennis ball as it leaves the sling is small compared to the maximum height reached during the flight.

With these assumptions, the equation can be solved for the time of flight

$$t_{\text{flight}} = \frac{2 v_{y_init}}{a_c}$$

Using Mathcad, the time of flight is predicted to be 1.9 seconds.

$$\begin{aligned}
 a_c &:= -g \quad \ll \text{the constant acceleration in this problem is due to gravity} \\
 t_{\text{flight}} &:= \frac{2 v_{y_init}}{a_c} \\
 t_{\text{flight}} &= 1.9 \text{ s}
 \end{aligned}$$

We can calculate the maximum height to see if the assumption that y_0 is negligible is reasonable. The maximum height occurs at $\frac{1}{2}$ the flight time (if $y_0 = y_{\text{final}}$, and air resistance is negligible).

$$\begin{aligned}
 y_0 &:= 0 \text{ m} \\
 y_{\text{fin}} &:= 0 \text{ m} \\
 y_{\text{max}} &:= y_0 + v_{y_init} \left(\frac{t_{\text{flight}}}{2} \right) + \frac{1}{2} a_c \left(\frac{t_{\text{flight}}}{2} \right)^2 \\
 y_{\text{max}} &= 4.4 \text{ m}
 \end{aligned}$$

Looking at the drawing of the slingshot, the ball is released at a height about two or three times the ball diameter, around 15 to 20 cm. 20 cm is nearly 5% of y_{\max} . That is probably negligible, but we can use Mathcad's iterative solver to find t_{flight} including $y_0 = 20$ cm.

$$\begin{aligned}
 y_0 &:= 20 \text{ cm} \\
 y_{\text{fin}} &:= 0 \text{ m} \\
 t_{\text{flight}} &:= 1.92 \text{ sec} \\
 &\text{given} \\
 y_{\text{fin}} &= y_0 + v_{y_{\text{init}}} t_{\text{flight}} + \frac{1}{2} a_c t_{\text{flight}}^2 \\
 t_{\text{flight}} &:= \text{find}(t_{\text{flight}}) \\
 t_{\text{flight}} &= 1.9232 \text{ s}
 \end{aligned}$$

Accounting for the initial height increased the predicted flight time from 1.902 to 1.923 seconds (about 1% difference).

Finally, we calculate the predicted horizontal travel distance.

$$\begin{aligned}
 x_0 &:= 0 \text{ m} \\
 x &:= x_0 + v_x t_{\text{flight}} \\
 x &= 38.5 \text{ m}
 \end{aligned}$$

Annotated Mathcad Worksheet

Projectile Motion: Slingshot Contest

Calculate Initial Velocity from Test Run

$$x_{\text{test}} := 60 \text{ mm} \quad \ll \text{horizontal travel distance for all four frames}$$

$$\theta := 25 \text{ deg}$$

$$d_{\text{test}} := \frac{x_{\text{test}}}{\cos(\theta)}$$

$$d_{\text{test}} = 66.2 \text{ mm} \quad \ll \text{distance ball travelled in direction of motion (four frames)}$$

$$\Delta t_{\text{pic}} := 0.0013 \text{ sec}$$

$$\Delta t_{\text{test}} := 3 \Delta t_{\text{pic}} \quad \ll 3 \text{ time intervals between the four photos}$$

$$v_{\text{init}} := \frac{d_{\text{test}}}{\Delta t_{\text{test}}}$$

$$v_{\text{init}} = 22.1 \frac{\text{m}}{\text{s}}$$

$$v_{y_init} := v_{\text{init}} \sin(\theta)$$

$$v_{y_init} = 9.3 \frac{\text{m}}{\text{s}} \quad \ll y \text{ component of initial velocity}$$

$$v_{x_init} := v_{\text{init}} \cos(\theta)$$

$$v_{x_init} = 20 \frac{\text{m}}{\text{s}}$$

$$v_x := v_{x_init} \quad \ll x \text{ component of velocity is constant (if air resistance is ignored)}$$

Calculate Time of Flight

The height of the tennis ball as it leaves the slingshot has been ignored in this calculation. This is a reasonable assumption if the initial height is small compared to the maximum height reached by the projectile.

$$y_0 := 0 \text{ m}$$

$$y_{\text{fin}} := 0 \text{ m} \quad \ll \text{assumes the ball is launched over a flat surface}$$

$$a_c := -g \quad \ll \text{the constant acceleration in this problem is due to gravity}$$

$$t_{\text{flight}} := \frac{2 v_{y_init}}{a_c}$$

$$t_{\text{flight}} = 1.902 \text{ s} \quad \ll \text{flight time calculated ignoring initial height}$$

$$y_{\text{max}} := y_0 + v_{y_init} t_{\text{flight}} + \frac{1}{2} a_c \left(\frac{t_{\text{flight}}}{2} \right)^2$$

$$y_{\text{max}} = 4.4 \text{ m}$$

Alternative Solution Method - Include Initial Height and Use Mathcad's Iterative Solver

$$y_0 := 20\text{m}$$

$$y_{\text{fin}} := 0\text{m}$$

$$t_{\text{flight}} := 1.93\text{sec} \quad \ll \text{ use the previously calculated value as an initial guess for the solver}$$

Given (The choice of guess value is not too critical,
but there is a second root at -0.02 seconds)

$$y_{\text{fin}} = y_0 + v_{y_init} t_{\text{flight}} + \frac{1}{2} a_c t_{\text{flight}}^2$$

$$t_{\text{flight}} := \text{Find}(t_{\text{flight}})$$

$$t_{\text{flight}} = 1.9232\text{s}$$

Calculate Horizontal Motion

$$x_0 := 0\text{m}$$

$$x := x_0 + v_x t_{\text{flight}}$$

$$x = 38.5\text{m}$$

Part 2. Solution

With the Mathcad worksheet, we can simply try some different angles until we calculate a predicted horizontal travel distance of 30 meters.

First, try 20 degrees.

Projectile Motion: Slingshot Contest

Calculate Initial Velocity Components

$$v_{\text{init}} := 22.1 \frac{\text{m}}{\text{s}}$$

$$\theta := 20\text{deg}$$

$$v_{y_init} := v_{\text{init}} \sin(\theta)$$

$$v_{y_init} = 7.6 \frac{\text{m}}{\text{s}}$$

$$v_{x_init} := v_{\text{init}} \cos(\theta)$$

$$v_{x_init} = 20.8 \frac{\text{m}}{\text{s}}$$

$$v_x := v_{x_init}$$

Calculate Time of Flight

$$a_c := -g$$

$$t_{\text{flight}} := \frac{2 v_{y_init}}{a_c}$$

$$t_{\text{flight}} = 1.5\text{s}$$

Calculate Horizontal Motion

$$x_0 := 0\text{m}$$

$$x := x_0 + v_x t_{\text{flight}}$$

$$x = 32\text{m}$$

With a little trial and error, the predicted angle should be 18.5 degrees.

Projectile Motion: Slingshot Contest

Calculate Initial Velocity Components

$$v_{\text{init}} := 22.1 \frac{\text{m}}{\text{s}}$$

$$\theta := 18.5 \text{ deg}$$

$$v_{y_init} := v_{\text{init}} \sin(\theta)$$

$$v_{y_init} = 7 \frac{\text{m}}{\text{s}}$$

$$v_{x_init} := v_{\text{init}} \cos(\theta)$$

$$v_{x_init} = 21 \frac{\text{m}}{\text{s}}$$

$$v_x := v_{x_init}$$

Calculate Time of Flight

$$a_c := -g$$

$$t_{\text{flight}} := \frac{2 v_{y_init}}{a_c}$$

$$t_{\text{flight}} = 1.4 \text{ s}$$

Calculate Horizontal Motion

$$x_0 := 0 \text{ m}$$

$$x := x_0 + v_x t_{\text{flight}}$$

$$x = 30 \text{ m}$$

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Curvilinear Motion: Normal and Tangential Components

Ref: Hibbeler § 12.7, Bedford & Fowler: Dynamics § 2.3

When the path of a particle is known, an n-t coordinate system with an origin at the location of the particle (at an instant in time) can be helpful in describing the motion of the particle. Hibbeler gives a concise procedure for analysis in section 12.7, which we will apply to the following example.

Example: A skateboarder in a quarter pipe.

A skateboarder has dropped into a 3 meter (radius) quarter pipe. When she's dropped a distance of 1 meter her speed is 3.2 m/s and is increasing by 7.8 m/s².

Determine the normal and tangential components of the skateboarder's acceleration.

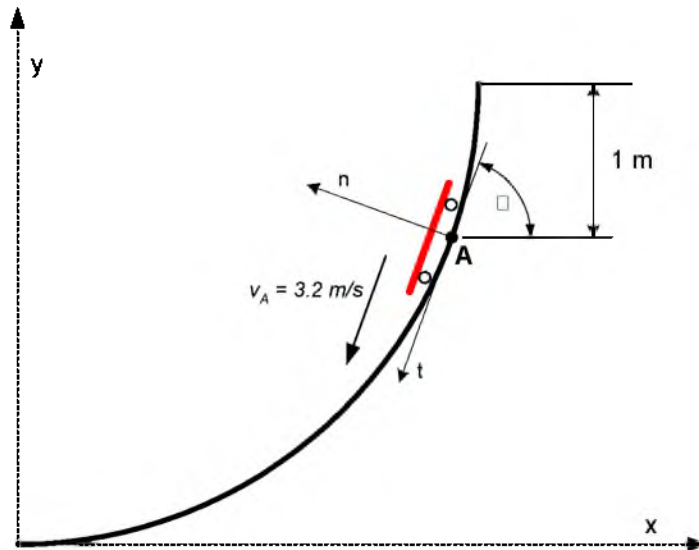
Solution

The equation of the skateboarder's path is that of a quarter circle:

$$y = R - \sqrt{R^2 - x^2}$$

where R is the radius, 3 m.

First, we need to determine the actual location of the skateboarder when she has dropped 1 m.



She is obviously $R - 1 = 2$ m off the ground, and her x-position can be determined from the path equation using either Mathcad's iterative solver (given/find), or symbolic math capability. Here, we use a given/find solve block.

```

R := 3m
y := 2m
guess:    x := 0m

Given

      y = R - √(R² - x²)

x := Find(x)

x = 2.828m

```

So, at the instant of interest, the skateboarder is located at $x = 2.828 \text{ m}$, $y = 2 \text{ m}$.

Next, the equation of the slope of the path at this point can be determined using Mathcad's derivative operator as follows:

```

Y(X) := R - √(R² - X²)

d
— Y(X) = —————
dX          1
          (9m² - X²)^(1/2)

```

Note: The path equation was written using upper-case variables (X and Y) because the lowercase variables (x and y) have already been assigned values. If the lower-case variables had been used, the symbolic math processor would have used the value of x to interpret the numeric value of the derivative instead of returning the equation for the derivative.

By assigning X a value, the numeric value of the derivative can be calculated.

```

X := 2.828m

d
— Y(X) = 2.825
dX

```

The angle at point A can then be determined as:

```

θ := atan(2.825)

θ = 70.507deg

```

The skateboarder's acceleration can be written in terms of normal and tangential velocities, as

$$\begin{aligned}
 \mathbf{a} &= \dot{v} \mathbf{u}_t + \frac{v^2}{\rho} \mathbf{u}_n \\
 &= 7.8 \frac{\text{m}}{\text{s}^2} \mathbf{u}_t + \frac{\left(3.2 \frac{\text{m}}{\text{s}}\right)^2}{\rho} \mathbf{u}_n
 \end{aligned}$$

The radius of curvature, ρ , of the path at the point of interest ($x=2.828 \text{ m}$, $y = 2 \text{ m}$) can be calculated as:

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{3}{2}}}{\left| \frac{d^2y}{dx^2} \right|}$$

In Mathcad, this equation is evaluated as follows:

$$\begin{aligned} Y(X) &:= R \sqrt{R^2 - X^2} \\ X &:= 2.828 \text{ m} \\ \rho &:= \frac{\left[1 + \left(\frac{d}{dX} Y(X) \right)^2 \right]^{\frac{3}{2}}}{\left| \frac{d^2}{dX^2} Y(X) \right|} \\ \rho &= 3 \text{ m} \end{aligned}$$

The calculated radius of curvature is 3 m – which should come as no surprise since the path is a circle of radius 3 m.

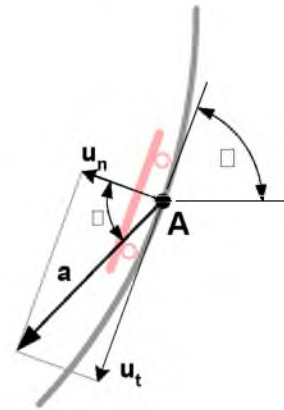
The equation for the acceleration of the skateboarder in terms of tangential and normal components is now:

$$\begin{aligned} \mathbf{a} &= \dot{v} \mathbf{u}_t + \frac{v^2}{\rho} \mathbf{u}_n \\ &= 7.8 \frac{\text{m}}{\text{s}^2} \mathbf{u}_t + \frac{\left(3.2 \frac{\text{m}}{\text{s}} \right)^2}{3 \text{ m}} \mathbf{u}_n \end{aligned}$$

The magnitude of the acceleration is found with the following calculation:

$$\begin{aligned} a &:= \sqrt{\left(7.8 \frac{\text{m}}{\text{s}^2} \right)^2 + \left[\frac{\left(3.2 \frac{\text{m}}{\text{s}} \right)^2}{3 \text{ m}} \right]^2} \\ a &= 8.514 \frac{\text{m}}{\text{s}^2} \end{aligned}$$

While the angle, ϕ , is found using the atan() function.



$$\phi := \text{atan} \left[\frac{\frac{7.8 \frac{\text{m}}{\text{s}^2}}{\text{s}}}{\frac{\left(3.2 \frac{\text{m}}{\text{s}}\right)^2}{3 \text{m}}} \right]$$

$$\phi = 66.365 \text{deg}$$

The angle of the acceleration from the positive x axis is:

$$\text{Angle} := \phi + 90 \text{deg} + \phi$$

$$\text{Angle} = 226.872 \text{deg}$$

Annotated Mathcad Worksheet

Acceleration of a Skateboarder in a Quarter Pipe

First, determine her position in x,y coordinates

$$R := 3 \text{m}$$

$$y := 2 \text{m}$$

$$\text{guess: } x := 0 \text{m}$$

Given

$$y = R - \sqrt{R^2 - x^2}$$

$$x := \text{Find}(x)$$

$$x = 2.828 \text{m}$$

The iterative solver was used here, but you could also have Mathcad solve directly for x using the symbolic math processor.

Switched to capital X and Y because I want the symbolic math processor to treat them as variables, and lower-case x and y have already been assigned values in this problem.

The path equation is defined.

$$Y(X) := R - \sqrt{R^2 - X^2}$$

The derivative of the path equation will be used to determine the slope at the point of interest.

$$\frac{d}{dX} Y(X) = \frac{1}{\left(9 \text{m}^2 - X^2\right)^{\frac{1}{2}}} X$$

The x value at the point of interest is specified to determine the numeric value of the derivative at that point.

$$X := 2.828 \text{ m}$$

$$\frac{d}{dX} Y(X) = 2.825$$

The slope is used to determine the angle of the path at the point of interest.

$$\square := \text{atan}(2.825)$$

$$\square = 70.507 \text{ deg}$$

The radius of curvature at the point of interest is calculated.

Note: This is done simply to show how Mathcad can calculate these derivatives - the radius of curvature is constant at 3 m for this path.

$$\square := \frac{\left[1 + \left(\frac{d}{dX} Y(X) \right)^2 \right]^{\frac{3}{2}}}{\left| \frac{d^2}{dX^2} Y(X) \right|}$$

$$\square = 3 \text{ m}$$

The magnitude of the acceleration is calculated.

$$a := \sqrt{\left(7.8 \frac{\text{m}}{\text{s}^2} \right)^2 + \left[\frac{\left(3.2 \frac{\text{m}}{\text{s}} \right)^2}{3 \text{ m}} \right]^2}$$

$$a = 8.514 \frac{\text{m}}{\text{s}^2}$$

The angle of the acceleration (relative to the t,n coordinate system) is calculated.

$$\square := \text{atan} \left[\frac{7.8 \frac{\text{m}}{\text{s}^2}}{\frac{\left(3.2 \frac{\text{m}}{\text{s}} \right)^2}{3 \text{ m}}} \right]$$

$$\square = 66.365 \text{ deg}$$

The angle of the acceleration relative to positive x axis is calculated.

$$\text{Angle} := \square + 90 \text{ deg} + \square$$

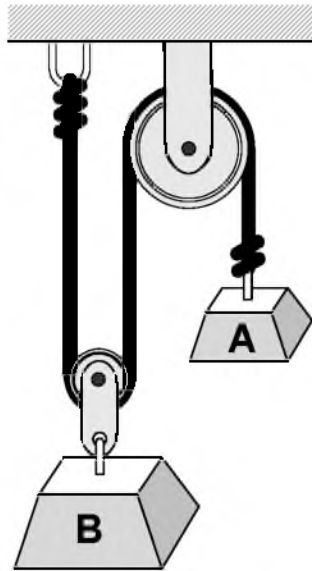
$$\text{Angle} = 226.872 \text{ deg}$$

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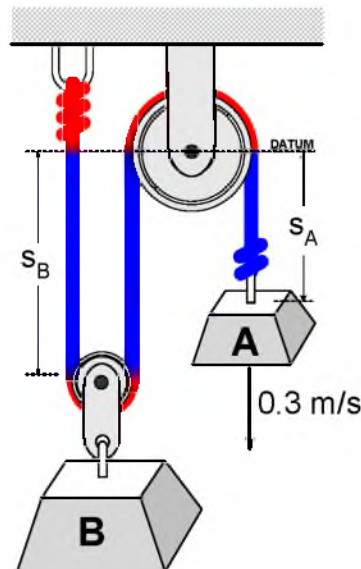
Dependent Motion of Two Particles

Ref: Hibbeler § 12.9, Bedford & Fowler: Dynamics, chapter 2 (no directly analogous section)

Sometimes the motion of one particle depends on the motion of another – this is called *dependent motion*. One situation that creates dependent motion is when two particles are connected by a cord around a pulley.



If weight A is pulled downward, weight B will be raised, but the total length of the cord, L_{total} , (assumed inextensible) is constant. The total length of the cord can be described in terms of cord lengths that change, s_A and s_B , and unchanging lengths, combined and called L_{const} (shown in red in the next figure.)



$$2s_B + s_A + L_{\text{const}} = L_{\text{total}}$$

Differentiating this equation with respect to time yields the time rates of change of position, or velocities of each weight.

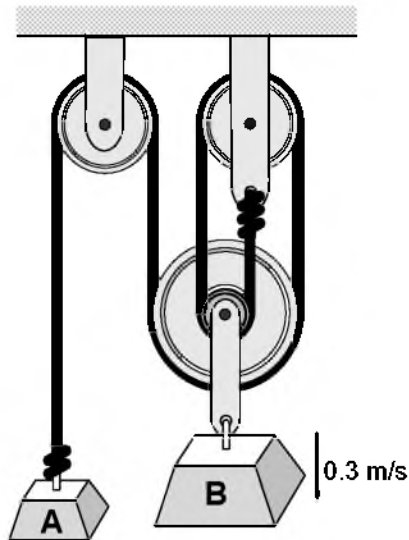
$$2 \frac{ds_B}{dt} + \frac{ds_A}{dt} + 0 = 0$$

$$2 v_B = -v_A$$

So, if weight A moves down at 0.3 m/s, weight B will move up at 0.15 m/s.

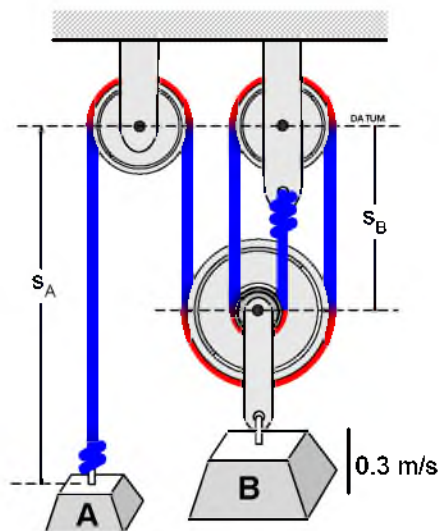
Example: Four Pulley System

At what rate, and in which direction must weight A move if weight B is to fall at a rate of 0.3 m/s?



Solution

There is a single cord between weights A and B, so the motion of one of the weights is dependent upon the other. While the physical system is somewhat more complex than the original, 2-pulley example, the solution is not very different. The total length of the cord is made up of 9 segments, as shown in the figure below. There are 4 equivalent-length sections that lengthen as weight B is lowered. The length of each of these sections is labeled s_B .



The length of the cord segment connected to weight A is labeled s_A . There are four cord segments around the four pulleys that do not change as the weights move. These segments are shown in red in the figure, and their lengths are combined and called L_{const} . The total cord length, then, is

$$4 s_B + s_A + L_{\text{const}} = L_{\text{total}}$$

Differentiating this equation with respect to time yields the time rates of change of position, or velocities of each weight.

$$4 \frac{ds_B}{dt} + \frac{ds_A}{dt} + 0 = 0$$

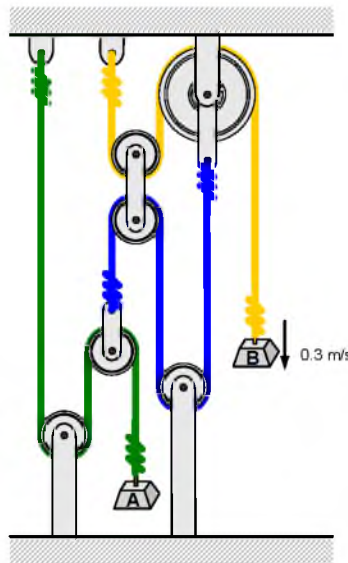
$$4 v_B = -v_A$$

So, if weight B moves down at 0.3 m/s, weight A will move up at 1.2 m/s.

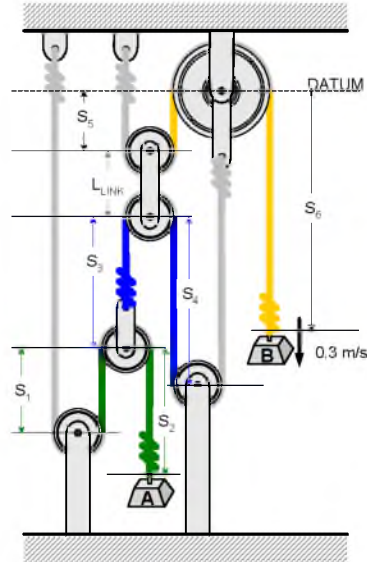
So far, these problems require only simple calculus, and there is little need for Mathcad's calculational abilities. However, when there are several cords involved, Mathcad's matrix math capabilities can come into play.

Example: Three Cord System

In the example shown below, three cords are connected around a complicated pulley system. The length of each of the cords is constant, so the sum of the segment lengths can be written for each cord.



To simplify things a bit, we first determine which segments are of constant length, shown in gray in the following figure. Then the segments that are changing length are labeled (s_1 through s_6 , as shown in the figure.)



The following relationships involving the changing segment lengths can be written:

- 1) Segments S_1 and S_2 , plus a fixed length, F_G , sum to the total length of the green cord, L_G .
- 2) Segments S_3 and S_4 , plus a fixed length, F_B , sum to the total length of the blue cord, L_B .
- 3) Segments S_5 and S_6 , plus a fixed length, F_Y , sum to the total length of the yellow cord, L_Y .

Two additional relationships can be written:

- 4) Segments S_1, S_3, S_5 and the length of the floating link, L_{LINK} , sum to an unknown, but constant (fixed) length, F_1 .
- 5) Segments S_4, S_5 and the length of the floating link, L_{LINK} , sum to an unknown, but constant (fixed) length, F_2 .

In summary, the following five equations are written:

$$\begin{aligned} S_1 + S_2 + F_G &= L_G \\ S_3 + S_4 + F_B &= L_B \\ S_5 + S_6 + F_Y &= L_Y \\ S_1 + S_3 + S_5 + L_{LINK} &= F_1 \\ S_4 + S_5 + L_{LINK} &= F_2 \end{aligned}$$

Taking time derivatives, the segment lengths become velocities, and the constant terms go to zero.

$$\begin{aligned} v_1 + v_2 + 0 &= 0 \\ v_3 + v_4 + 0 &= 0 \\ v_5 + v_6 + 0 &= 0 \\ v_1 + v_3 + v_5 + 0 &= 0 \\ v_4 + v_5 + 0 &= 0 \end{aligned}$$

Since v_6 is known to be 0.3 m/s (downward), these 5 equations in 5 unknowns can be solved. Including the known value, the equations become...

$$\mathbf{v}_4 + \mathbf{v}_5 = \mathbf{0}$$

$$C := \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad r := \begin{pmatrix} 0 \\ 0 \\ 0.3 \\ 0 \\ 0 \end{pmatrix} \frac{m}{s}$$
$$C_{\text{inv}} := C^{\square 1} \qquad v := C_{\text{inv}} \mathbb{I}^{\square}$$

$$C_{\text{inv}} = \begin{pmatrix} 0 & \square 1 & \square 2 & 1 & 1 \\ 1 & 1 & 2 & \square 1 & \square 1 \\ 0 & 1 & 1 & 0 & \square 1 \\ 0 & 0 & \square 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0.6 \\ \square 0.6 \\ \square 0.3 \\ 0.3 \\ \square 0.3 \end{pmatrix} \frac{\text{m}}{\text{s}}$$

$$L_A = S_5 + L_{LINK} + S_3 + S_2$$

The velocity of point A relative to the datum is the time derivative of L_A .

$$\begin{aligned} v_A &= \frac{dL_A}{dt} \\ &= \frac{dS_5}{dt} + 0 + \frac{dS_3}{dt} + \frac{dS_2}{dt} \\ &= v_5 + 0 + v_3 + v_2 \end{aligned}$$

Velocities v_2 , v_3 , and v_5 are known, so v_A can be determined. (Note: The vector index ORIGIN in Mathcad was set to 1 for this problem.)

$$\begin{aligned} v_A &:= v_5 + 0 + v_3 + v_2 \\ v_A &= 1.2 \frac{\text{m}}{\text{s}} \end{aligned}$$

Since the velocity at point B was 0.3 m/s downward, this result indicates that point A is moving at 1.2 m/s upward.

Annotated Mathcad Worksheet

Note: ORIGIN = 1 has been set using Math/Options for this problem.

Declare the coefficient matrix and right-hand-side vector

$$C := \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad r := \begin{pmatrix} 0 \\ 0 \\ -0.3 \\ 0 \\ 0 \end{pmatrix} \frac{\text{m}}{\text{s}}$$

Invert the coefficient matrix.

$$C_{\text{inv}} := C^{-1} \quad C_{\text{inv}} = \begin{pmatrix} 0 & -1 & -2 & 1 & 1 \\ 1 & 1 & 2 & -1 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Multiply the inverted coefficient matrix and the right-hand-side vector to calculate the segment velocities.

$$v := C_{\text{inv}} r \quad v = \begin{pmatrix} 0.6 \\ -0.6 \\ -0.3 \\ 0.3 \\ -0.3 \end{pmatrix} \frac{\text{m}}{\text{s}}$$

Use the segment velocities to calculate the velocity of point A relative to the datum.

$$\begin{aligned} v_A &:= v_5 + 0 + v_3 + v_2 \\ v_A &= 1.2 \frac{\text{m}}{\text{s}} \end{aligned}$$

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Kinetics of a Particle: Force and Acceleration

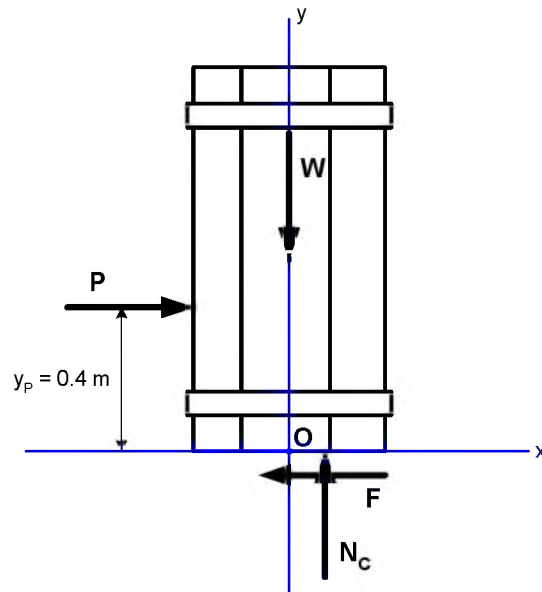
Ref: Hibbeler § 13.4, Bedford & Fowler: Dynamics § 3.1-3.4

In an earlier example (#8) we looked at “pushing a crate” and investigated the effect of changing the height of the push and the magnitude of the push on the rigid body. Specifically, we tested to see if the crate would tip or slip. In this example, we will determine the initial acceleration of the crate and the velocity after a short time at that acceleration.

Example: Pushing a Crate

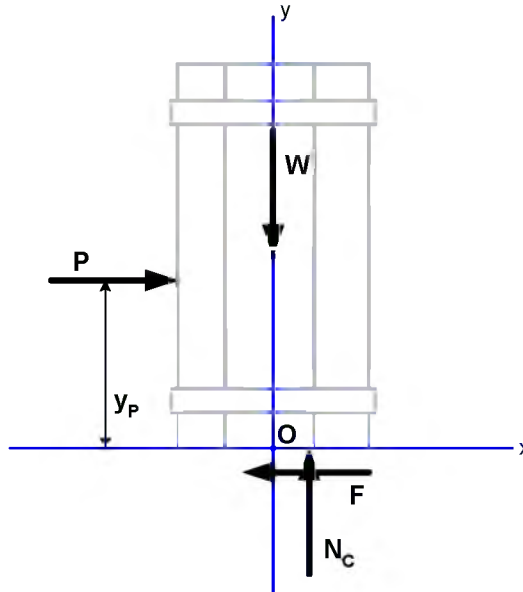
A crate weighs 35 kg, and is 0.5 m wide and 1.0 m in height. A force, P , is applied to the side of the crate at height $y_P = 0.4$ m from the floor. From the earlier example (#8) we know that this force is sufficient to overcome friction, and when applied at this height will not tip the crate. The coefficient of kinetic friction is 0.24.

Determine the initial acceleration and, assuming the acceleration remains constant, the expected velocity of the crate after 3 seconds.



Solution

First, a free-body diagram is drawn.



We start the solution by assigning values given in the problem statement to variables:

$$\begin{array}{l} M := 35 \text{ kg} \quad \mu_k := 0.24 \quad P := 120 \text{ N} \quad y_P := 0.4 \text{ m} \end{array}$$

Then we calculate the weight of the crate.

$$\begin{array}{l} W := M \cdot g \\ W = 343.2 \text{ N} \end{array}$$

Note: The acceleration due to gravity, g , is a predefined variable in Mathcad.

We can use the equation of motion for the y-components of force to determine the resultant normal force, N_C (but the result is not much of a surprise.)

$$\begin{array}{l} W + N_C = 0 \\ \text{so...} \\ N_C := -W \\ N_C = 343.2 \text{ N} \quad \text{acts in the } +x \text{ direction} \end{array}$$

Knowing N_C , the friction force can be determined.

$$\begin{array}{l} F := \mu_k \cdot N_C \quad \text{this equation does not account for direction} \\ F := -F \quad \text{sign changed since } F \text{ acts in } -x \text{ direction} \\ F = 82.4 \text{ N} \end{array}$$

Next, we use the equations of motions with the information on the free-body diagram to solve for the acceleration, a_x .

$$P + F = M a_x$$

so...

$$a_x := \frac{P + F}{M}$$

$$a_x = 1.075 \frac{\text{m}}{\text{s}^2}$$

Now the velocity after 3 seconds can be determined.

$$v_0 := 0 \frac{\text{m}}{\text{s}} \quad \Delta t := 3 \text{ s} \quad a := a_x$$

$$v := v_0 + a \Delta t$$

$$v = 3.225 \frac{\text{m}}{\text{s}}$$

Annotated Mathcad Worksheet

Pushing a Crate - Initial Acceleration

Data from the problem statement...

$$M := 350 \text{ kg} \quad \mu_k := 0.24 \quad P := 120 \text{ N} \quad y_P := 0.4 \text{ m}$$

Calculate the weight of the crate.

$$W := M g$$

$$W = 343.2 \text{ N}$$

EQ of MOTION: y components of force.

$$W + N_C = 0$$

so...

$$N_C := -W$$

$$N_C = 343.2 \text{ N} \quad \text{acts in the } +x \text{ direction}$$

Calculate the friction force, F.

$$F := \mu_k N_C \quad \text{this equation does not account for direction}$$

$$F := -F \quad \text{sign changed since } F \text{ acts in } -x \text{ direction}$$

$$F = 82.4 \text{ N}$$

EQ of MOTION: x components of force.

$$P + F = M a_x$$

so...

$$a_x := \frac{P + F}{M}$$

$$a_x = 1.075 \frac{\text{m}}{\text{s}^2}$$

Calculate the velocity after three seconds.

$$v_0 := 0 \frac{\text{m}}{\text{s}} \quad \Delta t := 3 \text{ s} \quad a := a_x$$

$$v := v_0 + a \Delta t$$

$$v = 3.225 \frac{\text{m}}{\text{s}}$$

Equations of Motion: Normal and Tangential Components

Ref: Hibbeler § 13.5, Bedford & Fowler: Dynamics § 3.4

In an earlier example (#14) we looked at the motion of a skateboarder along a curved path. Now, we apply the equations of motion to that problem to investigate the significance of friction in the problem.

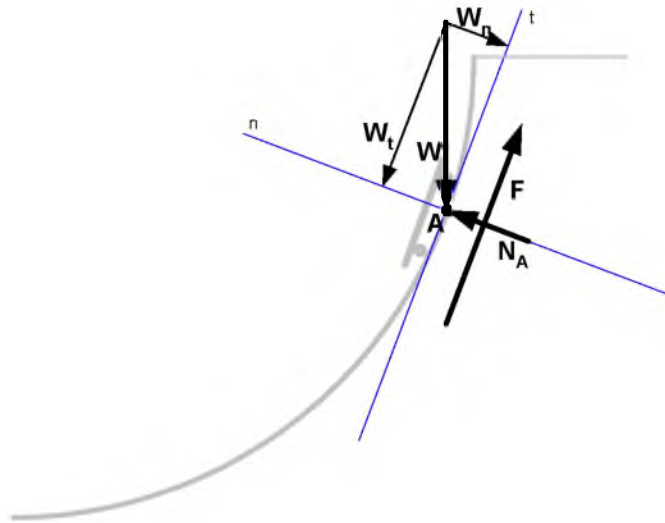
Example: A skateboarder in a quarter pipe.

A skateboarder with a mass of 55 kg has dropped into a 3 meter (radius) quarter pipe. When she's dropped a distance of 1 meter her speed is 3.2 m/s and is increasing by 7.8 m/s^2 . At this location the surface is at an angle of 19.4° from vertical.

Draw a free body diagram of this system including a friction force. Neglect the size of the skateboarder and her board. Solve the equations of motion for the normal force, N_A , and the friction force, F . Then, calculate the coefficient of kinetic friction required to generate the observed friction force.

Solution

A free-body diagram for this system looks like the following:



First, information from the problem statement is entered into a Mathcad worksheet.

$v := 3.2 \frac{\text{m}}{\text{s}}$	$a_t := 7.8 \frac{\text{m}}{\text{s}^2}$	$M := 55 \text{ kg}$
$\theta := 19.4 \text{ deg}$	<< angle of surface from vertical	
$r := 3 \text{ m}$	<< circular path, constant r	

The tangential acceleration at this location was given in the problem statement, as $a_t = 7.8 \text{ m/s}^2$. The normal acceleration can be calculated from the stated velocity (3.2 m/s) and the radius of curvature (3 m).

$$a_n := \frac{v^2}{r}$$

$$a_n = 3.4 \frac{\text{m}}{\text{s}^2}$$

The normal component of the equation of motion can be used to find N_A , but first we need the normal component of the skateboarder's weight.

$$W := Mg \quad W = 539 \text{ N}$$

$$W_n := W \sin(\theta) \quad W_n = 179 \text{ N} \quad \ll \text{minus sign accounts for direction}$$

$$W_t := W \cos(\theta) \quad W_t = 509 \text{ N}$$

Then find the normal force, N_A .

$$N_A + W_n = Ma_n$$

so...

$$N_A := Ma_n - W_n$$

$$N_A = 366.89 \text{ N}$$

Next, use the tangential component of the equation of motion to determine the friction force.

$$W_t + F = Ma_t$$

so...

$$F := Ma_t - W_t$$

$$F = -79.7 \text{ N} \quad \ll \text{acts in -t direction}$$

The friction force, F , and the normal force, N_A , can be used together to calculate the apparent coefficient of kinetic friction for this problem.

$$\mu_k := \frac{|F|}{N_A} \quad \ll \text{need only the magnitude of } F \text{ here}$$

$$\mu_k = 0.22$$

Annotated Mathcad Worksheet

Friction Force on a Skateboarder

Set problem parameters as stated in the problem.

$$v := 3.2 \frac{\text{m}}{\text{s}} \quad a_t := 7.8 \frac{\text{m}}{\text{s}^2} \quad M := 55 \text{ kg}$$

$$\theta := 19.4 \text{ deg} \quad \ll \text{ angle of surface from vertical}$$

$$r := 3 \text{ m} \quad \ll \text{ circular path, constant } r$$

Solve for the normal acceleration.

$$a_n := \frac{v^2}{r}$$

$$a_n = 3.4 \frac{\text{m}}{\text{s}^2}$$

Find the normal and tangential components of the skateboarder's weight.

$$W := Mg \quad W = 539 \text{ N}$$

$$W_n := W \sin(\theta) \quad W_n = 179 \text{ N} \quad \ll \text{ minus sign accounts for direction}$$

$$W_t := W \cos(\theta) \quad W_t = 509 \text{ N}$$

EQ of MOTION: normal component

$$N_A + W_n = Ma_n$$

so...

$$N_A := Ma_n - W_n$$

$$N_A = 366.89 \text{ N}$$

EQ of MOTION: tangential component

$$W_t + F = Ma_t$$

so...

$$F := Ma_t - W_t$$

$$F = -79.7 \text{ N} \quad \ll \text{ acts in } -t \text{ direction}$$

Calculate the coefficient of kinetic friction.

$$\mu_k := \frac{|F|}{N_A} \quad \ll \text{ need only the magnitude of } F \text{ here}$$

$$\mu_k = 0.22$$

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Principle of Work and Energy

Ref: Hibbeler § 14.3, Bedford & Fowler: Dynamics § 8.1-8.2

The *principle of work and energy* states that the work done by all of the external forces and couples as a rigid body moves between positions 1 and 2 is equal to the change in the body's potential energy. Hibbeler writes the resulting equation as

$$T_1 + \sum U_{1 \rightarrow 2} = T_2 \quad \text{Hibbeler (14-7)}$$

The same equation is written as

$$U_{12} = T_2 - T_1 \quad \text{Bedford and Fowler: Dynamics (8.4)}$$

by Bedford and Fowler.

The difference between these two equations is simply nomenclature. Both $\sum U_{1 \rightarrow 2}$ and U_{12} represent the sum of work done by all external forces and couples on the body. We can use the principle of work and energy to solve problems involving force, displacement, and velocity.

Example: Skid-to-Stop Braking Distance

The driver of a 1600 kg passenger car hits the brakes and skids to a stop to try to avoid hitting a deer. When the deer suddenly appeared in the headlights [at an estimated distance of 300 ft (91 m)] the car was moving at 75 mph (120 km/hr).

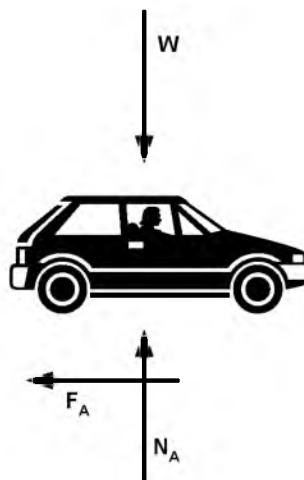
After the incident long skid marks were visible on the pavement, and the deer was seen running over a nearby hill. Did the driver get the car stopped in time, or did the deer manage to get out of the way?

Assumptions:

- Assume a 1.5 second reaction time – that is, it took 1.5 seconds from the time the driver saw the deer until he or she pressed the brake pedal.
- Assume a dry, flat road with a coefficient of friction, $\mu_k = 0.80$.

Solution

A free body diagram of this system shows the various forces acting on the car.



Because the road was assumed to be flat, the normal force, N_A , is equal to the vehicle's weight.

$$\begin{aligned} M &:= 1600 \text{ kg} \\ W &:= M \cdot g \\ N_A &:= W \\ N_A &= 15691 \text{ N} \end{aligned}$$

The friction force during the skid (wheels locked) is calculated using the normal force and the coefficient of kinetic friction, α_k .

$$\begin{aligned} \alpha_k &:= 0.80 \\ F_A &:= \alpha_k \cdot N_A \\ F_A &= 12553 \text{ N} \end{aligned}$$

The length of the skid can be determined using the principle of work and energy.

$$\begin{aligned} T_1 + \sum U_{1 \rightarrow 2} &= T_2 \\ \frac{1}{2} m v_1^2 + \sum U_{1 \rightarrow 2} &= \frac{1}{2} m v_2^2 \end{aligned}$$

The final velocity, v_2 , is zero (skid-to-stop), and the only force acting on the car during the deceleration is the friction force, F_A , acting through the skid distance, s .

$$\frac{1}{2} m v_1^2 + (-F_A s) = 0$$

This equation can be solved for the skid distance.

$$\begin{aligned} v_1 &:= 75 \frac{\text{mi}}{\text{hr}} \\ s &:= \frac{\frac{1}{2} M v_1^2}{F_A} \\ s &= 72 \text{ m} \end{aligned}$$

That looks good; it is possible to stop in less than 91 meters, the initial distance between the car and the deer. But, we also need to account for the distance the car traveled in the 1.5 seconds between the time the driver saw the deer and the moment the brakes were applied.

$$\begin{aligned} t_{\text{reaction}} &:= 1.5 \text{ sec} \\ s_{\text{reaction}} &:= v_1 \cdot t_{\text{reaction}} \\ s_{\text{reaction}} &= 50 \text{ m} \end{aligned}$$

The total distance traveled between the moment the deer was spotted and the vehicle coming to a complete stop is $50 + 72 = 122 \text{ m}$.

$$s_{\text{total}} := s + s_{\text{reaction}}$$

$$s_{\text{total}} = 122\text{m}$$

The driver did not get stopped within 91 m. The deer had to get out of the way in order to escape injury.

Annotated Mathcad Worksheet

Skid-to-Stop Calculations

Calculate the magnitude of the normal force.

$$M := 1600\text{kg}$$

$$W := Mg$$

$$N_A := W$$

$$N_A = 15691\text{N}$$

Calculate the friction force.

$$\mu_k := 0.80$$

$$F_A := \mu_k N_A$$

$$F_A = 12553\text{N}$$

Calculate the skid distance.

$$v_1 := 75 \frac{\text{mi}}{\text{hr}}$$

$$s := \frac{\frac{1}{2} M v_1^2}{F_A}$$

$$s = 72\text{m}$$

Calculate the distance traveled during the driver's reaction time.

$$t_{\text{reaction}} := 1.53\text{sec}$$

$$s_{\text{reaction}} := v_1 t_{\text{reaction}}$$

$$s_{\text{reaction}} = 50\text{m}$$

Calculate the total distance traveled after spotting the deer.

$$s_{\text{total}} := s + s_{\text{reaction}}$$

$$s_{\text{total}} = 122\text{m}$$

Rotation About a Fixed Axis

Ref: Hibbeler § 16.3, Bedford & Fowler: Dynamics § 9.1

Because drive motors are routinely used, solving problems dealing with rotation about fixed axes is commonplace. The example used here looks at a very old-fashioned drive motor – a water wheel.

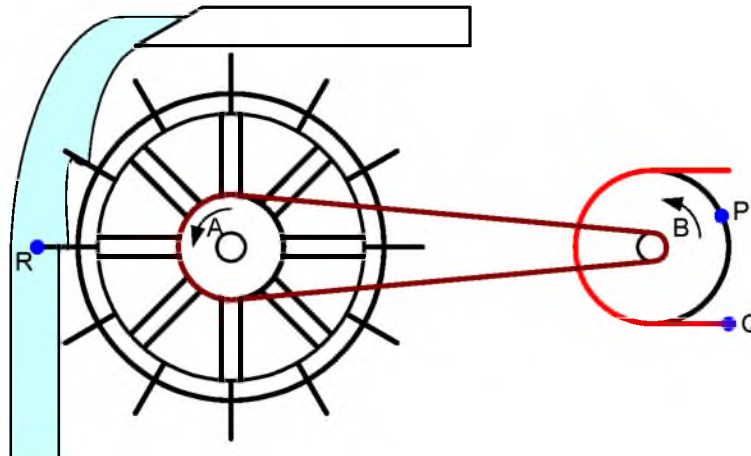
Example: Water Wheel

Long ago, a water wheel was used to drive a mill. The point labeled “R” on the outermost edge of the water wheel was 1.25 m from the center of the wheel. A drive wheel, labeled “A”, was directly attached to the water wheel. A belt connected the drive wheel to the shaft on wheel “B”. The dimensions of the various wheels and shafts are listed below.

Shortly after the water gate was opened to allow water to flow over the wheel, then water wheel had an angular acceleration of 0.1 rad/s^2 . Assume the belts did not slip.

Determine:

- The magnitudes of the velocities and accelerations at points R and P after the water wheel has made two complete revolutions.
- The velocity and acceleration of point Q on the belt leaving wheel B.



<i>Dimensions</i>	<i>Diameter</i>
Wheel A	0.7 m
Wheel B	1 m
Shaft B	0.19 m

Solution, part a (point R)

Movement through a complete circle is equivalent to moving a point through 2π radians, so two complete revolutions of point R would move that point through 4π radians, or $4 \times 3.1416 = 12.57$ radians. So,

$$\theta_R = 12.57 \text{ rad}$$

The water wheel is accelerating at constant rate, $\alpha_A = 0.1 \text{ rad/s}^2$, so the angular velocity of point R is

$$\begin{aligned}
 \omega_R^2 &= \omega_0^2 + 2\omega_A (\theta_R - \theta_0) \\
 &= 0 + 2(0.1 \text{ rad/s}^2)(12.57 \text{ rad}) \\
 &= 2.51 \text{ rad}^2/\text{s}^2 \\
 \omega_R &= 1.59 \text{ rad/s}
 \end{aligned}$$

In Mathcad, this calculation can be performed as follows:

$$\begin{aligned}
 \omega_A &:= 0.1 \frac{\text{rad}}{\text{s}^2} & \omega_0 &:= 0 \frac{\text{rad}}{\text{s}} & \theta_0 &:= 0 \text{ rad} \\
 \theta_R &:= 4 \text{ rad} \\
 \omega_R &:= \sqrt{\omega_0^2 + 2\omega_A (\theta_R - \theta_0)} \\
 \omega_R &= 1.59 \frac{\text{rad}}{\text{s}}
 \end{aligned}$$

Note: Units were used on the zero terms. This is required or Mathcad will detect an error when attempting to subtract the θ values. In order to perform the subtraction, the variables must have the same units.

The velocity at point R can now be determined.

$$\begin{aligned}
 r_R &:= 1.25 \text{ m} \\
 v_R &:= \omega_R r_R & v_R &= 1.98 \frac{\text{m}}{\text{s}}
 \end{aligned}$$

Point R has tangential and normal components of acceleration. They are calculated as follows.

$$\begin{aligned}
 a_{R_t} &:= \omega_A r_R & a_{R_t} &= 0.125 \frac{\text{m}}{\text{s}^2} \\
 a_{R_n} &:= \omega_R^2 r_R & a_{R_n} &= 3.142 \frac{\text{m}}{\text{s}^2}
 \end{aligned}$$

Then the magnitude of the acceleration at point R can be determined.

$$\begin{aligned}
 a_R &:= \sqrt{a_{R_t}^2 + a_{R_n}^2} \\
 a_R &= 3.144 \frac{\text{m}}{\text{s}^2}
 \end{aligned}$$

Solution, part a (point P)

The connector between drive wheel A and wheel B is the belt between the two wheels. If the belt does not slip, then

$$s = \omega_A r_A = \omega_B r_{B_shaft}$$

Notice that the radius of the shaft on wheel B is used in this calculation, because the belt from wheel A goes around the shaft on wheel B, not wheel B itself. However, the angular displacement of the shaft on wheel B is the same as the angular displacement of wheel B – they are directly connected.

$$\begin{aligned}\omega_A &:= \omega_R & \omega_A &= 12.57 \text{ rad} \\ r_A &:= 0.35 \text{ m} \\ r_{B_shaft} &:= \frac{0.19}{2} \text{ m} \\ \omega_B &:= \frac{\omega_A r_A}{r_{B_shaft}} & \omega_B &= 46.3 \text{ rad}\end{aligned}$$

Since the belt connecting wheel A and shaft B has the same speed and tangential component of acceleration, we can write:

$$\begin{aligned}v &= \omega_A r_A = \omega_B r_{B_shaft} \\ a_t &= \omega_A r_A = \omega_B r_{B_shaft}\end{aligned}$$

These relationships can be used to find the angular velocity and tangential component of acceleration of shaft B, which are also the angular velocity and tangential component of acceleration of point P on wheel B.

$$\begin{aligned}\omega_A &:= \omega_R \\ \omega_B &:= \frac{\omega_A r_A}{r_{B_shaft}} & \omega_B &= 5.84 \frac{\text{rad}}{\text{s}} \\ \omega_B &:= \frac{\omega_A r_A}{r_{B_shaft}} & \omega_B &= 0.37 \frac{\text{rad}}{\text{s}^2}\end{aligned}$$

Now it is possible to calculate the magnitudes of the velocity and acceleration at point P.

$$\begin{aligned}r_P &:= 0.5 \text{ m} \\ v_P &:= \omega_B r_P & v_P &= 2.92 \frac{\text{m}}{\text{s}} \\ a_{P_t} &:= \omega_B r_P & a_{P_t} &= 0.184 \frac{\text{m}}{\text{s}^2} \\ a_{P_n} &:= \omega_B^2 r_P & a_{P_n} &= 17.1 \frac{\text{m}}{\text{s}^2} \\ a_P &:= \sqrt{a_{P_t}^2 + a_{P_n}^2} \\ a_P &= 17.1 \frac{\text{m}}{\text{s}^2}\end{aligned}$$

Solution, part b (point Q)

Point Q has the same velocity and tangential component of acceleration as point P.

$$v_Q = 2.92 \text{ m/s}$$

$$a_Q = 0.184 \text{ m/s}^2$$

Annotated Mathcad Worksheet

Water Wheel

Calculate the angular velocity of the water wheel (and point R).

$$\omega_A := 0.1 \frac{\text{rad}}{\text{s}^2} \quad \omega_0 := 0 \frac{\text{rad}}{\text{s}} \quad \omega_0 := 0 \text{ rad}$$

$$\omega_R := 4 \pi \text{ rad}$$

$$\omega_R := \sqrt{\omega_0^2 + 2 \omega_A (\omega_R - \omega_0)}$$

$$\omega_R = 1.59 \frac{\text{rad}}{\text{s}}$$

Find the velocity at point R.

$$r_R := 1.25 \text{ m}$$

$$v_R := \omega_R r_R \quad v_R = 1.98 \frac{\text{m}}{\text{s}}$$

Find the tangential and angular components of acceleration at point R.

$$a_{R_t} := \omega_A r_R \quad a_{R_t} = 0.125 \frac{\text{m}}{\text{s}^2}$$

$$a_{R_n} := \omega_R^2 r_R \quad a_{R_n} = 3.142 \frac{\text{m}}{\text{s}^2}$$

Determine the magnitude of the acceleration at point R.

$$a_R := \sqrt{a_{R_t}^2 + a_{R_n}^2}$$

$$a_R = 3.144 \frac{\text{m}}{\text{s}^2}$$

Determine the angular displacement of wheel B.

$$\theta_A := \theta_R \quad \theta_A = 12.57 \text{ rad}$$

$$r_A := 0.35 \text{ m}$$

$$r_{B_shaft} := \frac{0.19}{2} \text{ m}$$

$$\theta_B := \frac{\theta_A r_A}{r_{B_shaft}} \quad \theta_B = 46.3 \text{ rad}$$

Determine the angular velocity and tangential component of acceleration of shaft B (and point P).

$$\omega_A := \omega_R$$

$$\omega_B := \frac{\omega_A r_A}{r_{B_shaft}} \quad \omega_B = 5.84 \frac{\text{rad}}{\text{s}}$$

$$a_{B_t} := \frac{\omega_A^2 r_A}{r_{B_shaft}} \quad a_{B_t} = 0.37 \frac{\text{rad}}{\text{s}^2}$$

Find the velocity and acceleration at point P.

$$r_P := 0.3 \text{ m}$$

$$v_P := \omega_B r_P \quad v_P = 2.92 \frac{\text{m}}{\text{s}}$$

$$a_{P_t} := \omega_B^2 r_P \quad a_{P_t} = 0.184 \frac{\text{m}}{\text{s}^2}$$

$$a_{P_n} := \omega_B^2 r_P \quad a_{P_n} = 17.1 \frac{\text{m}}{\text{s}^2}$$

$$a_P := \sqrt{a_{P_t}^2 + a_{P_n}^2}$$

$$a_P = 17.1 \frac{\text{m}}{\text{s}^2}$$

Find the velocity and acceleration of point Q.

$$v_Q := v_P \quad v_Q = 2.92 \frac{\text{m}}{\text{s}}$$

$$a_Q := a_{P_t} \quad a_Q = 0.184 \frac{\text{m}}{\text{s}^2}$$